

Asymptotic Consistency for Nonconvex Risk-Averse Stochastic Optimization with Infinite Dimensional Decision Spaces

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Abstract. Optimal values and solutions of empirical approximations of stochastic optimization problems can be viewed as statistical estimators of their true values. From this perspective, it is important to understand the asymptotic behavior of these estimators as the sample size goes to infinity. This area of study has a long tradition in stochastic programming. However, the literature is lacking consistency analysis for problems in which the decision variables are taken from an infinite dimensional space, which arise in optimal control, scientific machine learning, and statistical estimation. By exploiting the typical problem structures found in these applications that give rise to hidden norm compactness properties for solution sets, we prove consistency results for nonconvex risk-averse stochastic optimization problems formulated in infinite dimensional space. The proof is based on several crucial results from the theory of variational convergence. The theoretical results are demonstrated for several important problem classes arising in the literature.

Key words. asymptotic consistency, empirical approximation, sample average approximation, Monte Carlo sampling, risk-averse optimization, PDE-constrained optimization, uncertainty quantification, stochastic programming

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1 Introduction

The asymptotic behavior of empirical approximations is a central point of study in optimization under uncertainty. There is a long tradition going back to the fundamental contributions [31, 22, 62, 57, 63, 33, 64, 49, 65, 55, 50, 40, 69, 56]. These works have since given rise to standard derivation techniques for problems with finite dimensional decision spaces. There are in essence three main techniques used to obtain asymptotic statements. The first possibility uses epi-convergence of sample-based approximations of objective functions over compact sets and therefore draws from powerful statements in the theory of variational convergence. The second type of method employs a uniform law of large numbers for sample-based approximations of objective functions. Finally, asymptotic statements can also be derived from stability estimates for optimal values and solutions with respect to probability semimetrics. This requires, amongst other things, that the class of integrands in the objective constitutes a P -uniformity class for the semimetric in question.

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Given a general stochastic optimization problem

$$\min_{z \in Z_{\text{ad}}} \mathbb{E}_P[F(z)], \quad (1)$$

an empirical approximation would take the form

$$\min_{z \in Z_{\text{ad}}} \mathbb{E}_{P_N}[F(z)], \quad (2)$$

where the original probability measure P is replaced by a (sequence of) typically discrete approximation(s) P_N for $N \in \mathbb{N}$. For example, the probability measure P_N could be an empirical probability measure associated with a random sample of size N from P . This is a common approach often referred to as “sample average approximation” (SAA), see e.g., [34, 68]. A data-driven viewpoint can be drawn from machine learning in which (1) represents the “population risk minimization” problem and (2) the corresponding “empirical risk minimization” problem. Here, the underlying probability measure of the data P is typically unknown. It is therefore of interest to understand the behavior of solutions in the big data limit (as $N \rightarrow \infty$).

The main questions can be easily stated: Do the optimal values and solution sets of (2) converge to their “true” counterparts for (1) as N passes to infinity and what is the strongest form of stochastic convergence that can be guaranteed? If we treat the N -dependent objects as statistical estimators of the true values and seek to prove at least convergence in probability, then these are questions of consistency, cf. [66].

Motivated by recent advances in partial differential equation (PDE)-constrained optimization under uncertainty [24, 37], scientific machine learning [9, 48], nonconvex stochastic programming [42, 51, 16], and statistical estimation [59, 60, 43], we provide such consistency results for stochastic optimization problems in which the decision variables z may be taken in an infinite dimensional space Z . We will consider more general “risk-averse” problems in which the expectation \mathbb{E}_P is allowed to be replaced by certain classes of convex risk functionals \mathcal{R} . And as it is often lacking in the application areas mentioned above, we do not assume convexity of the integrand F . For consistency results on finite dimensional risk-averse stochastic optimization problems, we refer the reader to [17, 66, 68].

From an abstract perspective, we consider stochastic optimization problems of the type

$$\min_{z \in Z_{\text{ad}}} \mathcal{R}[F(z)] + \wp(z). \quad (3)$$

Here, Z_{ad} is typically a closed convex subset of an infinite dimensional space; \wp is a deterministic convex cost function; F is a random integrand that typically depends on the solution of a differential equation subject to random inputs; and \mathcal{R} is a convex functional that acts as a numerical surrogate for our risk preference, e.g., a convex combination of $\mathbb{E}_P[X]$ and a semideviation $\mathbb{E}_P[\max\{0, X - \mathbb{E}_P[X]\}]$.

Despite the past successes in consistency analysis listed above, there is a major difficulty in extending the finite dimensional arguments to the infinite dimensional setting. In order to use both the epigraphical as well as the uniform law of large numbers approaches, we need an appropriately defined *norm* compact set that contains both the approximate N -dependent solutions as well as true solutions. It is not enough for the feasible set to be closed and bounded. For example, the simple set of pointwise bilateral constraints

$$Z_{\text{ad}} := \{ z \in L^2(0, 1) : 0 \leq z(x) \leq 1 \text{ for a.e. } x \in (0, 1) \}$$

is weakly sequentially compact in $L^2(0, 1)$, but not norm compact. The literature is not void of results for infinite dimensional problems. However, the stability statements developed in [30, 58] and

the large deviation-type bounds derived in [46, 45] have only been demonstrated for strongly convex risk-neutral problems. While it may be possible to extend some of these results to a risk-averse setting, it appears rather challenging to obtain statements about the consistency of minimizers without strong convexity. In the recent preprint [44], consistency results for optimal values and solutions are established for risk-neutral PDE-constrained optimization using a uniform law of large numbers. Our framework and that in [44] are different. Besides considering risk-neutral problems, i.e., $\mathcal{R} = \mathbb{E}_P$, the work [44] requires the integrands be continuously differentiable, the decision space be a separable Hilbert space, and requires a specific strongly convex control regularization in the objective function. Moreover, [44] establishes consistency for exact solutions while we are able to establish consistency for approximate solutions using epiconvergence.

The paper is structured as follows. In Section 2, we introduce the basic notation, assumptions, and several preliminary results necessary for the remaining parts of the text. Afterwards, in Section 3, we present our main result. Finally, the utility of the main consistency result is demonstrated for several problem classes in Section 4.

2 Notation, Assumptions, and Preliminary Results

We introduce several concepts, notation and assumptions that are required in the text.

2.1 Probability and Function Spaces

Throughout the text, all spaces are defined over the real numbers \mathbb{R} and metric spaces are equipped with their Borel σ -field. Let Ξ be a complete separable metric space, \mathcal{A} the associated Borel σ -algebra, and $\mathbb{P} : \mathcal{A} \rightarrow [0, 1]$ a probability measure. The triple $(\Xi, \mathcal{A}, \mathbb{P})$ is always assumed to be a complete probability space. Throughout the manuscript, (Ω, \mathcal{F}, P) is a probability space.

If Υ is a Banach space, then its topological dual space is denoted by Υ^* . Their dual pairing is denoted by $\langle v, w \rangle_{\Upsilon^*, \Upsilon}$ for $v \in \Upsilon^*$, $w \in \Upsilon$. If Υ is reflexive, we identify its bi-dual $(\Upsilon^*)^*$ with Υ . Throughout the text, we will use $p \in [1, \infty)$ for a general integrability exponent. In the application section, we will consider problems involving random partial differential equations. These require several function spaces. The underlying physical domain $D \subset \mathbb{R}^d$ with $d \in \{1, 2, 3\}$ will always be an open bounded Lipschitz domain.

For a Banach space $(V, \|\cdot\|_V)$ we will denote the Lebesgue–Bochner space $L^p(\Xi, \mathcal{A}, \mathbb{P}; V)$ of all strongly \mathcal{A} -measurable V -valued functions by

$$L^p(\Xi, \mathcal{A}, \mathbb{P}; V) = \{u : \Xi \rightarrow V : u \text{ strongly } \mathcal{A}\text{-measurable and } \|u\|_{L^p(\Xi, \mathcal{A}, \mathbb{P}; V)} < \infty\}$$

endowed with the natural norms $\|u\|_{L^p(\Xi, \mathcal{A}, \mathbb{P}; V)} = (\mathbb{E}_{\mathbb{P}}[\|u\|_V^p])^{1/p}$ for $p \in [1, \infty)$ and for bounded fields: $\|u\|_{L^p(\Xi, \mathcal{A}, \mathbb{P}; V)} = \mathbb{P}\text{-ess sup}_{\xi \in \Xi} \|u(\xi)\|_V$. In the event that $V = \mathbb{R}$, we simply write $L^p(\Xi, \mathcal{A}, \mathbb{P})$.

For the PDE applications, we use $L^p(D)$ to denote the usual Lebesgue space of p -integrable (or essentially bounded) functions over D . For more details on Lebesgue–Bochner spaces, we refer the reader to [29, Chapter III]. We denote convergence in the norm by \rightarrow and weak convergence by \rightharpoonup . For a sequence (v_k) , we denote by $(v_k)_K$ a subsequence of (v_k) , where $K \subset \mathbb{N}$ is an infinite index set.

Given two random variables $X_1, X_2 \in L^p(\Omega, \mathcal{F}, P)$ for $p \in [1, \infty)$, we say that X_1 and X_2 are *distributionally equivalent* with respect to P if $P(X_1 \leq t) = P(X_2 \leq t)$ for all $t \in \mathbb{R}$. A functional $\rho : L^p(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ is said to be *law invariant* with respect to P if for all distributionally equivalent random variables $X_1, X_2 \in L^p(\Omega, \mathcal{F}, P)$ we have $\rho(X_1) = \rho(X_2)$. In this setting, it therefore makes

sense to use the (abuse of) notation $\rho(H_X)$, where $H_X(t) := P(X \leq t)$ with $t \in \mathbb{R}$ as opposed to $\rho(X)$. We caution that this does not mean we redefine the function ρ over a space of càdlàg functions. For a (cumulative) distribution function H defined on \mathbb{R} , its quantile function H^{-1} is defined by $H^{-1}(t) := \inf_{s \in \mathbb{R}} \{s : H(s) \geq t\}$ for $t \in (0, 1)$. Let (Ω, \mathcal{F}, P) be nonatomic and let $\rho : L^p(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ be law invariant for $p \in [1, \infty)$. Since (Ω, \mathcal{F}, P) is nonatomic, there exists a random variable $G : \Omega \rightarrow [0, 1]$ with uniform distribution ν on $[0, 1]$ [21, Prop. A.7] (see also [11, Prop. 9.1.11]). Let $X \in L^p(\Omega, \mathcal{F}, P)$ be a random variable. Since $H_X^{-1}(G(\cdot))$ has the same distribution function as that of X [20, Prop. 9.1.2] and $H_X^{-1}(G(\cdot)) \in L^p(\Omega, \mathcal{F}, P)$, we have $\rho(H_X) = \rho(H_X^{-1}(G(\cdot)))$. More generally, we write $\rho(H)$ instead of $\rho(H^{-1}(G(\cdot)))$, provided that H is a distribution function on \mathbb{R} with $\int_0^1 |H^{-1}(q)|^p d\nu(q) < \infty$.

2.2 Convex Analysis and Several Key Functionals

Given a Banach space V , the (effective) domain of an extended real-valued function $f : V \rightarrow (-\infty, \infty]$, will be denoted by $\text{dom}(f) := \{x \in V : f(x) < \infty\}$. We typically exclude convex functions that take the value $-\infty$. For $f : V \rightarrow (-\infty, \infty]$ and $\varepsilon > 0$, $x_\varepsilon \in V$ is an ε -minimizer of f provided $\inf_{v \in V} f(v)$ is finite and $f(x_\varepsilon) \leq \inf_{v \in V} f(v) + \varepsilon$. The ε -solution set ($\varepsilon \geq 0$) is then the set $\mathcal{S}^\varepsilon := \{x \in V : f(x) \leq \inf_{v \in V} f(v) + \varepsilon\}$, provided that $\inf_{v \in V} f(v)$ is finite. We use the convention $\mathcal{S} = \mathcal{S}^0$.

Let Υ be a normed space. For $x \in \Gamma \subset \Upsilon$ and $\Psi \subset \Upsilon$, we define

$$\text{dist}(x, \Psi) = \inf_{y \in \Psi} \|x - y\|_\Upsilon \quad \text{and} \quad \mathbb{D}(\Gamma, \Psi) = \sup_{x \in \Gamma} \text{dist}(x, \Psi).$$

We recall that a Banach space V has the *Radon–Riesz (Kadec–Klee) property* if $v_k \rightarrow v$ whenever $(v_k) \subset V$ is a sequence with $v_k \rightarrow v \in V$ and $\|v_k\|_V \rightarrow \|v\|_V$ as $k \rightarrow \infty$. More generally, we will say that a function $\varphi : V \rightarrow [0, \infty)$ is an *R-function* if it is convex and continuous, and if $v_k \rightarrow v$ as $k \rightarrow \infty$ whenever $(v_k) \subset V$ is a sequence with $v_k \rightarrow v \in V$ and $\varphi(v_k) \rightarrow \varphi(v)$ as $k \rightarrow \infty$. Notions related to but different from that of an R-function are available in the literature, such as functions having the Kadec property and strongly rotund functions [13, 14]. The notion of an R-function is first introduced in our manuscript. If V is a reflexive Banach space, then there exists an R-function on V [13, p. 154]. A notion of regularizers different from that of an R-function can be found in [32].

As the following fact demonstrates, the class of R-functions is rather large and includes, e.g., typical cost functions and regularizers used in PDE-constrained optimization. See Section 4.2 for an example of an R-function in the context of PDE-constrained optimization.

Lemma 1. *Let V be a Banach space. If $\wp : [0, \infty) \rightarrow [0, \infty)$ is convex and strictly increasing and $\varphi : V \rightarrow [0, \infty)$ is an R-function, then $\wp \circ \varphi$ is an R-function. In particular, if V has the Radon–Riesz property, then $\wp \circ \|\cdot\|_V$ is an R-function.*

Proof. The function $\wp \circ \varphi$ is convex and continuous. Let $v_k \rightarrow v$ and $\wp(\varphi(v_k)) \rightarrow \wp(\varphi(v))$. Since \wp is strictly increasing on $[0, \infty)$, it has a continuous inverse. Hence $\varphi(v_k) \rightarrow \varphi(v)$. \square

For a Banach space V and a complete probability space $(\Xi, \mathcal{A}, \mathbb{P})$, $f : V \times \Xi \rightarrow (-\infty, \infty]$ is said to be *random lower semicontinuous* provided f is jointly measurable (with respect to the tensor-product σ -algebra of the Borel σ -algebra on V and \mathcal{A}) and $f(\cdot, \xi)$ is lower semicontinuous for every $\xi \in \Xi$. If Υ_1 and Υ_2 are metric spaces, then $G : \Upsilon_1 \times \Xi \rightarrow \Upsilon_2$ is a *Carathéodory mapping* provided $G(v, \cdot)$ is measurable for all $v \in \Upsilon_1$ and $G(\cdot, \xi)$ is continuous for all $\xi \in \Xi$.

Finally, there are many concepts of risk measures in the literature. We will work with the following with further refinements as needed in the text below. Let $\rho : L^p(\Omega, \mathcal{F}, P) \rightarrow (-\infty, \infty]$. We consider the following conditions on the functional ρ .

- (R1) Convexity. For all $X, Y \in L^p(\Omega, \mathcal{F}, P)$ and $\lambda \in (0, 1)$, we have $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$.
- (R2) Monotonicity. For all $X, Y \in L^p(\Omega, \mathcal{F}, P)$ such that $X \leq Y$ w.p. 1, we have $\rho(X) \leq \rho(Y)$.
- (R3) Translation equivariance. If $X \in L^p(\Omega, \mathcal{F}, P)$ and C is a degenerate random variable with $C = c$ w.p. 1 for some $c \in \mathbb{R}$, then $\rho(X + C) = \rho(X) + c$.
- (R4) Positive homogeneity. If $X \in L^p(\Omega, \mathcal{F}, P)$ and $\gamma > 0$, then $\rho(\gamma X) = \gamma\rho(X)$.

The risk measure ρ is called *convex* if it satisfies (R1)–(R3) and it is referred to as *coherent* if it satisfies (R1)–(R4), see [2], [23], and in particular [68, p. 231].

2.3 Epiconvergence and Weak Inf-Compactness

Variational convergence, in particular (Mosco-)epiconvergence, plays a central role in consistency analysis. We provide here the necessary definitions and results from the literature. In addition, we prove several new results that are tailored to applications involving PDEs with random inputs.

We recall the notions of epiconvergence and Mosco-epiconvergence [4, 19].

Definition 2 (Epiconvergence). *Let V be a complete metric space. Let $\phi_k: V \rightarrow (-\infty, \infty]$ be a sequence and let $\phi: V \rightarrow (-\infty, \infty]$ be a function. The sequence (ϕ_k) epiconverges to ϕ if for each $v \in V$*

1. *and each $(v_k) \subset V$ with $v_k \rightarrow v$ as $k \rightarrow \infty$, $\liminf_{k \rightarrow \infty} \phi_k(v_k) \geq \phi(v)$, and*
2. *there exists $(v_k) \subset V$ with $v_k \rightarrow v$ as $k \rightarrow \infty$ such that $\limsup_{k \rightarrow \infty} \phi_k(v_k) \leq \phi(v)$.*

In many instances in infinite dimensional optimization, especially the calculus of variations, optimal control, and PDE-constrained optimization we are forced to work with weaker topologies in the context of variational convergence. If the underlying space is a reflexive Banach space, then we may appeal to epiconvergence in the sense of Mosco, which was introduced in [47].

Definition 3 (Mosco-epiconvergence). *Let V be a reflexive Banach space and let $V_0 \subset V$ be a closed, nonempty, convex set. Let $\phi_k: V_0 \rightarrow (-\infty, \infty]$ be a sequence and let $\phi: V_0 \rightarrow (-\infty, \infty]$ be a function. The sequence (ϕ_k) Mosco-epiconverges to ϕ if for each $v \in V_0$*

1. *and each $(v_k) \subset V_0$ with $v_k \rightarrow v$ as $k \rightarrow \infty$, $\liminf_{k \rightarrow \infty} \phi_k(v_k) \geq \phi(v)$, and*
2. *there exists $(v_k) \subset V_0$ with $v_k \rightarrow v$ such that $\limsup_{k \rightarrow \infty} \phi_k(v_k) \leq \phi(v)$.*

In the definition of Mosco-epiconvergence, we allow for the sequence (ϕ_k) and the epi-limit ϕ to be defined on a nonempty, convex, closed subset of a reflexive Banach space. This allows us to model constraints without the need for indicator functions. We will see below in Theorem 4 that this variation on the original definition leaves the crucial implications of Mosco-epiconvergence intact. In other words, Theorem 4 provides conditions sufficient for consistency of optimal values of Mosco-epiconvergent objective functions; compare with [3, Thm. 1.10], [18, Thm. 5.3], and [14, Thm. 6.2.8], for example.

Theorem 4. *Let V be a reflexive Banach space and let $V_0 \subset V$ be a closed, nonempty, convex set. Suppose that $h_k: V_0 \rightarrow (-\infty, \infty]$ Mosco-epiconverges to $h: V_0 \rightarrow (-\infty, \infty]$. Let $(v_k) \subset V_0$ and $(\varepsilon_k) \subset [0, \infty)$ be sequences such that $\varepsilon_k \rightarrow 0^+$ and for each $k \in \mathbb{N}$, let v_k satisfy*

$$h_k(v_k) \leq \inf_{v \in V_0} h_k(v) + \varepsilon_k.$$

If $(v_k)_K$ is a subsequence of (v_k) such that $v_k \rightarrow \bar{v}$ as $K \ni k \rightarrow \infty$, then

1. $\bar{v} \in V_0$,
2. $h(\bar{v}) = \inf_{v \in V_0} h(v)$,

3. $\inf_{v \in V_0} h_k(v) \rightarrow \inf_{v \in V_0} h(v)$ as $K \ni k \rightarrow \infty$,
4. $h_k(v_k) \rightarrow h(\bar{v})$ as $K \ni k \rightarrow \infty$.

Proof. Since $(v_k) \subset V_0$ and V_0 is weakly sequentially closed, we have $\bar{v} \in V_0$. Since (h_k) Mosco-epiconverges to h on V_0 , it epiconverges to h , where V_0 may be understood as a complete metric space using the norm topology. Hence

$$\limsup_{k \rightarrow \infty} \inf_{v \in V_0} h_k(v) \leq \inf_{v \in V_0} h(v); \quad (4)$$

see, e.g., [3, Props. 1.14 and 2.9]. Then Mosco-epiconvergence ensures

$$\liminf_{K \ni k \rightarrow \infty} \inf_{v \in V_0} h_k(v) = \liminf_{K \ni k \rightarrow \infty} [\varepsilon_k + \inf_{v \in V_0} h_k(v)] \geq \liminf_{K \ni k \rightarrow \infty} h_k(v_k) \geq h(\bar{v}).$$

Combined with (4), we find that $h(\bar{v}) = \inf_{v \in V_0} h(v)$ and $\inf_{v \in V_0} h_k(v) \rightarrow \inf_{v \in V_0} h(v)$ as $K \ni k \rightarrow \infty$. The assertion $h_k(v_k) \rightarrow h(\bar{v})$ as $K \ni k \rightarrow \infty$ is implied by the above derivations and

$$\limsup_{K \ni k \rightarrow \infty} h_k(v_k) \leq \limsup_{K \ni k \rightarrow \infty} [\varepsilon_k + \inf_{v \in V_0} h_k(v)] = \limsup_{K \ni k \rightarrow \infty} \inf_{v \in V_0} h_k(v) \leq \limsup_{k \rightarrow \infty} \inf_{v \in V_0} h_k(v).$$

□

Proposition 5 demonstrates a weak compactness property of approximate minimizers to “regularized” optimization problems with Mosco-epiconvergent objective functions. Let Z be a reflexive Banach space, let $Z_{\text{ad}} \subset Z$ be a closed, nonempty, convex set, and let $f_k, f : Z_{\text{ad}} \rightarrow (-\infty, \infty]$. Furthermore, let $\varphi : Z \rightarrow [0, \infty)$ be a convex, continuous function. We define the optimal values

$$\mathbf{m}_k^* := \inf_{z \in Z_{\text{ad}}} \{f_k(z) + \varphi(z)\} \quad \text{and} \quad \mathbf{m}^* := \inf_{z \in Z_{\text{ad}}} \{f(z) + \varphi(z)\} \quad (5)$$

and the solution sets

$$\mathcal{S}_k^{\varepsilon_k} := \{z \in Z_{\text{ad}} : f_k(z) + \varphi(z) \leq \mathbf{m}_k^* + \varepsilon_k\} \quad \text{and} \quad \mathcal{S} := \{z \in Z_{\text{ad}} : f(z) + \varphi(z) = \mathbf{m}^*\}.$$

Proposition 5. *Let Z be a reflexive Banach space, let $Z_{\text{ad}} \subset Z$ be a nonempty, closed, convex set, let $\varphi : Z \rightarrow [0, \infty)$ be a convex, continuous function, and let $Z_0 \subset Z$ be bounded. Suppose that $f_k : Z_{\text{ad}} \rightarrow (-\infty, \infty]$ Mosco-epiconverges to $f : Z_{\text{ad}} \rightarrow (-\infty, \infty]$. Let $(\varepsilon_k) \subset [0, \infty)$ be a sequence with $\varepsilon_k \rightarrow 0^+$. Suppose that $\mathcal{S} \neq \emptyset$ and that for all $k \in \mathbb{N}$,*

$$\mathcal{S}_k^{\varepsilon_k} \subset Z_0 \quad \text{and} \quad \mathcal{S}_k^{\varepsilon_k} \neq \emptyset.$$

If (z_k) is a sequence with $z_k \in \mathcal{S}_k^{\varepsilon_k}$ for all $k \in \mathbb{N}$ and $(z_k)_K$ is a subsequence of (z_k) , then $(z_k)_K$ has a further subsequence $(z_k)_{K_1}$ converging weakly to some $\bar{z} \in \mathcal{S}$ and $\varphi(z_k) \rightarrow \varphi(\bar{z})$ as $K_1 \ni k \rightarrow \infty$.

Proof. Since $(z_k)_K \subset Z_{\text{ad}}$, $(z_k)_K \subset Z_0$, Z_0 is bounded, and Z_{ad} is closed and convex, $(z_k)_K$ has a further subsequence $(z_k)_{K_1}$ such that $z_k \rightharpoonup \bar{z} \in Z_{\text{ad}}$ as $K_1 \ni k \rightarrow \infty$ [12, Thms. 2.23 and 2.28]. Since $\bar{z} \in Z_{\text{ad}}$, the Mosco-epiconvergence of (f_k) to f ensures the existence of a sequence $(\tilde{z}_k) \subset Z_{\text{ad}}$ such that $\tilde{z}_k \rightarrow \bar{z} \in Z_{\text{ad}}$ as $k \rightarrow \infty$ and $\limsup_{k \rightarrow \infty} f_k(\tilde{z}_k) \leq f(\bar{z})$. Since $\tilde{z}_k \rightarrow \bar{z}$ implies $\tilde{z}_k \rightharpoonup \bar{z}$, we have $\lim_{k \rightarrow \infty} f_k(\tilde{z}_k) = f(\bar{z})$. Since $z_k \in \mathcal{S}_k^{\varepsilon_k}$ and $\tilde{z}_k \in Z_{\text{ad}}$, we have for all $k \in \mathbb{N}$,

$$f_k(z_k) + \varphi(z_k) \leq f_k(\tilde{z}_k) + \varphi(\tilde{z}_k) + \varepsilon_k. \quad (6)$$

Since (f_k) Mosco-epiconverges to f , we have $f(\bar{z}) \leq \liminf_{K_1 \ni k \rightarrow \infty} f_k(z_k)$. Combined with the fact that φ is continuous and

$$\liminf_{K_1 \ni k \rightarrow \infty} f_k(z_k) + \limsup_{K_1 \ni k \rightarrow \infty} \varphi(z_k) \leq \limsup_{K_1 \ni k \rightarrow \infty} f_k(z_k) + \varphi(z_k),$$

the estimate (6) ensures

$$\begin{aligned} f(\bar{z}) + \limsup_{K_1 \ni k \rightarrow \infty} \varphi(z_k) &\leq \limsup_{K_1 \ni k \rightarrow \infty} f_k(\tilde{z}_k) + \varphi(\tilde{z}_k) + \varepsilon_k \leq \limsup_{k \rightarrow \infty} f_k(\tilde{z}_k) + \varphi(\tilde{z}_k) + \varepsilon_k \\ &= \lim_{k \rightarrow \infty} f_k(\tilde{z}_k) + \varphi(\tilde{z}_k) + \varepsilon_k = f(\bar{z}) + \varphi(\bar{z}). \end{aligned}$$

Since $z_k \rightarrow \bar{z}$ as $K_1 \ni k \rightarrow \infty$, $\mathcal{S} \neq \emptyset$, and (f_k) Mosco-epiconverges to f , Theorem 4 ensures $\bar{z} \in \mathcal{S}$. Since $\bar{z} \in \mathcal{S}$, we have $f(\bar{z}) \in \mathbb{R}$. Thus $\limsup_{K_1 \ni k \rightarrow \infty} \varphi(z_k) \leq \varphi(\bar{z})$. Since φ is convex and continuous, it is weakly lower semicontinuous. Combined with $z_k \rightarrow \bar{z}$ as $K_1 \ni k \rightarrow \infty$, we have $\varphi(z_k) \rightarrow \varphi(\bar{z})$ as $K_1 \ni k \rightarrow \infty$. \square

While the sum of an Mosco-epiconvergent sequence and a convex, continuous function Mosco-epiconverge, Proposition 5 allows us to draw further conclusions about the minimizers to composite optimization problems defined by sums of Mosco-epiconvergent and convex, continuous functions than a direct application of the ‘‘sum rule.’’ For example, if φ is an R-function, then the sequence $(z_k)_{K_1}$ considered in Proposition 5 converges strongly to an element of \mathcal{S} .

Corollary 6. *If the hypotheses of Proposition 5 hold true and φ is an R-function, then each subsequence of (z_k) has a further subsequence converging strongly to an element of \mathcal{S} .*

Remark 7. If Z_{ad} is bounded, then we can choose $Z_0 = Z_{\text{ad}}$ in Proposition 5. The condition $\mathcal{S}_k^{\varepsilon_k} \subset Z_0$ for all $k \in \mathbb{N}$ in Proposition 5 is related to a ‘‘weak inf-compactness’’ condition, provided that Z_0 is also convex and bounded. In this case, Z_0 is weakly (sequentially) compact. Instead of requiring $\mathcal{S}_k^{\varepsilon_k} \subset Z_0$ for all $k \in \mathbb{N}$, we could require for some $\gamma \in \mathbb{R}$ and for all $k \in \mathbb{N}$,

$$\emptyset \neq \{z \in Z_{\text{ad}} : f_k(z) + \varphi(z) \leq \gamma\} \subset Z_0. \quad (7)$$

The level set condition (7) ensures that $\mathcal{S}_k^{\varepsilon_k}$ is nonempty, provided that f_k is weakly lower semicontinuous. In case that Z_0 is norm compact, the condition (7) has been used, for example, in Theorem 2.1 in [40] to establish consistency properties for infinite dimensional stochastic programs. If $\sup_{k \in \mathbb{N}} \mathbf{m}_k^* < \infty$, $\gamma > \sup_{k \in \mathbb{N}} \mathbf{m}_k^*$ and for all $k \in \mathbb{N}$,

$$\{z \in Z_{\text{ad}} : f_k(z) + \varphi(z) \leq \gamma\} \subset Z_0,$$

then $\mathcal{S}_k^{\varepsilon_k} \subset Z_0$ for all sufficiently large $k \in \mathbb{N}$ since we eventually have $\sup_{k \in \mathbb{N}} \mathbf{m}_k^* + \varepsilon_k \leq \gamma$.

Corollary 8. *If the hypotheses of Proposition 5 hold, then $\mathbf{m}_k^* \rightarrow \mathbf{m}^*$ as $k \rightarrow \infty$. If furthermore φ is an R-function, then $\mathbb{D}(\mathcal{S}_k^{\varepsilon_k}, \mathcal{S}) \rightarrow 0$ as $k \rightarrow \infty$.*

Proof. Let $z_k \in \mathcal{S}_k^{\varepsilon_k}$ for each $k \in \mathbb{N}$. The hypotheses ensure that (f_k) Mosco-epiconverges to f . Let $(\mathbf{m}_k^*)_K$ be a subsequence of (\mathbf{m}_k^*) . Proposition 5 ensures that $(z_k)_K$ has a further subsequence $(z_k)_{K_1}$ that weakly converges to some element in \mathcal{S} . Combined with Theorem 4, we find that $\mathbf{m}_k^* \rightarrow \mathbf{m}^*$ as $K_1 \ni k \rightarrow \infty$. Since \mathcal{S} is nonempty, $\mathbf{m}^* \in \mathbb{R}$. Putting together the pieces, we have shown that each subsequence of (\mathbf{m}_k^*) has a further subsequence converging to \mathbf{m}^* . Hence $\mathbf{m}_k^* \rightarrow \mathbf{m}^*$ as $k \rightarrow \infty$.

It must still be shown that $\mathbb{D}(\mathcal{S}_k^{\varepsilon_k}, \mathcal{S}) \rightarrow 0$ as $k \rightarrow \infty$. Since $\mathcal{S}_k^{\varepsilon_k} \subset Z_0$ and $\mathcal{S} \subset Z$ are nonempty, and Z_0 is bounded, we have $\mathbb{D}(\mathcal{S}_k^{\varepsilon_k}, \mathcal{S}) \leq \mathbb{D}(Z_0, \mathcal{S}) < \infty$ for all $k \in \mathbb{N}$. Let us define

the sequence $\varrho_k := \mathbb{D}(\mathcal{S}_k^{\varepsilon_k}, \mathcal{S})$. Let $(\varrho_k)_K$ be a subsequence of (ϱ_k) . We have just shown that $\varrho_k \leq \mathbb{D}(Z_0, \mathcal{S}) < \infty$. Moreover $\varrho_k \geq 0$. Using the definition of the deviation, we find that there exists for each $k \in \mathbb{N}$, $\tilde{z}_k \in \mathcal{S}_k^{\varepsilon_k}$ such that $\varrho_k \leq \text{dist}(\tilde{z}_k, \mathcal{S}) + 1/k$. Corollary 6 ensures that $(\tilde{z}_k)_K$ has a further subsequence $(\tilde{z}_k)_{K_1}$ that strongly converges to some $\bar{z} \in \mathcal{S}$. Since \mathcal{S} is nonempty, $\text{dist}(\cdot, \mathcal{S})$ is (Lipschitz) continuous [1, Thm. 3.16]. It follows that $\text{dist}(\tilde{z}_k, \mathcal{S}) \rightarrow 0$ as $K_1 \ni k \rightarrow \infty$. Hence $\varrho_k \rightarrow 0$ as $K_1 \ni k \rightarrow \infty$. Since each subsequence of (ϱ_k) has a further subsequence converging to zero, $\varrho_k \rightarrow 0$ as $k \rightarrow \infty$. \square

Proposition 9 demonstrates that epiconvergence can imply Mosco-epiconvergence. This result is particularly relevant for PDE-constrained problems. If V is a Banach space and Y is a complete metric space, we refer to a mapping $G : V \rightarrow Y$ as *completely continuous* if $(v_k) \subset V$ and $v_k \rightarrow v \in V$ implies $G(v_k) \rightarrow G(v)$.

Proposition 9. *Let $Z_0 \subset Z$ be a nonempty, closed, convex subset of a reflexive Banach space Z and let $Y_0 \subset Y$ be a closed subset of a Banach space Y . Suppose that $\mathbf{B} : Z \rightarrow Y$ is linear and completely continuous with $\mathbf{B}(Z_0) \subset Y_0$. If $h_k : Y_0 \rightarrow (-\infty, \infty]$ epiconverges to $h : Y_0 \rightarrow (-\infty, \infty]$ and $h_k \circ \mathbf{B} : Z_0 \rightarrow (-\infty, \infty]$ epiconverges to $h \circ \mathbf{B} : Z_0 \rightarrow (-\infty, \infty]$, then $h_k \circ \mathbf{B} : Z_0 \rightarrow (-\infty, \infty]$ Mosco-epiconverges to $h \circ \mathbf{B} : Z_0 \rightarrow (-\infty, \infty]$.*

Proof. Fix $\bar{z} \in Z_0$. Let $(z_k) \subset Z_0$ be a sequence with $z_k \rightarrow \bar{z}$. We have $\mathbf{B}z_k, \mathbf{B}\bar{z} \in Y_0$. The complete continuity of \mathbf{B} yields $\mathbf{B}z_k \rightarrow \mathbf{B}\bar{z}$ as $k \rightarrow \infty$. Since (h_k) epiconverges to h , $\liminf_{k \rightarrow \infty} h_k(\mathbf{B}z_k) \geq h(\mathbf{B}\bar{z})$. The hypotheses ensure that $h_k \circ \mathbf{B}$ epiconverges to $h \circ \mathbf{B}$. Putting together the pieces, we conclude that $(h_k \circ \mathbf{B})$ Mosco-epiconverges to $h \circ \mathbf{B}$. \square

For the applications considered in Sections 4.2 and 4.3, \mathbf{B} is the adjoint operator of a compact (Sobolev) embedding operator, and hence linear and completely continuous.

3 Consistency of Empirical Approximations

We consider the potentially infinite dimensional risk-averse stochastic program

$$\min_{z \in Z_{\text{ad}}} \mathcal{R}[F(\mathbf{B}z)] + \wp(z), \quad (8)$$

where

$$F(y)(\omega) := f(y, \xi(\omega)), \quad (9)$$

the set Y_0 is a closed subset of a separable Banach space Y , $f : Y_0 \times \Xi \rightarrow \mathbb{R}$ is a Carathéodory function, and $\mathbf{B} : Z \rightarrow Y$ is a linear, continuous operator. Moreover $\xi : \Omega \rightarrow \Xi$ is a random element with law $\mathbb{P} = P \circ \xi^{-1}$, $f(y, \cdot) \in L^p(\Xi, \mathcal{A}, \mathbb{P})$, and $\mathcal{R} : L^p(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ with $1 \leq p < \infty$ is a law invariant convex risk measure.

We introduce the empirical approximation of (8) for the case when (Ω, \mathcal{F}, P) is nonatomic. Let ξ^1, ξ^2, \dots be defined on a complete probability space $(\Omega', \mathcal{F}', P')$ and assume the sequence is composed of independent identically distributed Ξ -valued random elements each with law $\mathbb{P} = P \circ \xi^{-1}$. For $y \in Y_0$ and $N \in \mathbb{N}$, the empirical distribution function $\hat{H}_{y,N}(\cdot; \omega')$ of the sample $f(y, \xi^1(\omega')), \dots, f(y, \xi^N(\omega'))$ is defined by

$$\hat{H}_{y,N}(t; \omega') := \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{(-\infty, t]}(f(y, \xi^i(\omega'))),$$

where $\omega' \in \Omega'$ and $\mathbf{1}_{(-\infty, t]}$ is the indicator function of the interval $(-\infty, t]$. We denote by $\hat{H}_{y,N}^{-1}(\cdot; \omega')$ the quantile function of $\hat{H}_{y,N}(\cdot; \omega')$. We often omit writing the second arguments of $\hat{H}_{y,N}$ and $\hat{H}_{y,N}^{-1}$. The empirical approximation of (8) is given by

$$\min_{z \in Z_{\text{ad}}} \mathcal{R}[\hat{H}_{\mathbf{B}z, N}] + \wp(z). \quad (10)$$

Recall from our discussion on law invariant risk measures that $\mathcal{R}[\hat{H}_{\mathbf{B}z, N}]$ means the risk measure \mathcal{R} does not distinguish between z, N -dependent random variables with distribution functions equivalent to $\hat{H}_{\mathbf{B}z, N}$.

Our consistency analysis is based on the conditions in Assumption 10.

- Assumption 10.**
1. The space Z is a separable, reflexive Banach space, $Z_{\text{ad}} \subset Z$ is nonempty, closed, convex and bounded. The space Y is a separable Banach space, and $Y_0 \subset Y$ is closed, and $p \in [1, \infty)$.
 2. The mapping $\mathbf{B} : Z \rightarrow Y$ is linear and completely continuous and $\mathbf{B}(Z_{\text{ad}}) \subset Y_0$.
 3. The function $\wp : Z \rightarrow [0, \infty)$ is convex and continuous.
 4. The function $f : Y_0 \times \Xi \rightarrow \mathbb{R}$ is a Carathéodory function.
 5. For all $y \in Y_0$, $f(y, \cdot) \in L^p(\Xi, \mathcal{A}, \mathbb{P})$ and $F : Y_0 \rightarrow L^p(\Omega, \mathcal{F}, P)$ defined in (9) is continuous.
 6. For each $\bar{y} \in Y_0$, there exists a neighborhood $\mathcal{V}_{\bar{y}} \subset Y_0$ of \bar{y} and a random variable $h \in L^p(\Xi, \mathcal{A}, \mathbb{P})$ such that $f(y, \cdot) \geq h(\cdot)$ for all $y \in \mathcal{V}_{\bar{y}}$.

Let m^* be the optimal value of problem (8) and let \mathcal{S} be its solution set. Furthermore, let \hat{m}_N^* be the optimal value of (10) and let $\hat{\mathcal{S}}_N^r$ be its set of r -minimizers, where $r \geq 0$. The “with probability one”-statements in Theorem 11 are with respect to P' .

Theorem 11. *Let Assumption 10 hold. Suppose further that (Ω, \mathcal{F}, P) is nonatomic and complete. Let $\mathcal{R} : L^p(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ be a convex, law invariant risk measure. If $(r_N) \subset [0, \infty)$ is a deterministic sequence such that $r_N \rightarrow 0$ as $N \rightarrow \infty$, then $\hat{m}_N^* \rightarrow m^*$ w.p. 1 as $N \rightarrow \infty$. If furthermore \wp is an R -function, then $\mathbb{D}(\hat{\mathcal{S}}_N^{r_N}, \mathcal{S}) \rightarrow 0$ w.p. 1 as $N \rightarrow \infty$.*

To establish Theorem 11, we verify the hypotheses of Corollaries 8 and 22 (in the Appendix).

Lemma 12. *If Assumption 10 holds and $\mathcal{R} : L^p(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ is a convex risk measure, then $Z_{\text{ad}} \ni z \mapsto \mathcal{R}[F(\mathbf{B}z)]$ is completely continuous.*

Proof. Since \mathcal{R} is a finite-valued convex risk measure, it is continuous [61, Cor. 3.1]. Assumption 10 ensures the continuity of $y \mapsto F(y)$. Hence $y \mapsto \mathcal{R}[F(y)]$ is continuous. Now the complete continuity of \mathbf{B} implies that of $z \mapsto \mathcal{R}[F(\mathbf{B}z)]$. \square

We recall from our discussion on law invariant risk measures in Section 2, the identity $\mathcal{R}[\hat{H}_{y,N}(\omega')] = \mathcal{R}[\hat{H}_{y,N}^{-1}(G(\cdot); \omega')]$, where $G : \Omega \rightarrow [0, 1]$ is a random variable with uniform distribution ν .

Lemma 13. *Let Assumption 10 hold. Suppose further that (Ω, \mathcal{F}, P) is nonatomic and complete. Let $\mathcal{R} : L^p(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ be a convex, law invariant risk measure. Then $Y_0 \times \Omega' \ni (y, \omega') \mapsto \mathcal{R}[\hat{H}_{y,N}^{-1}(G(\cdot); \omega')]$ is a Carathéodory function.*

Proof. Let $f(y, \xi^{(1)}) \leq \dots \leq f(y, \xi^{(N)})$ be the order statistics of the sample $f(y, \xi^1), \dots, f(y, \xi^N)$. For $q \in (0, 1]$ and $y \in Y_0$, we have $\hat{H}_{y,N}(q; \omega') = f(y, \xi^{(j)}(\omega'))$ if $q \in ((j-1)/N, j/N]$ irrespective of whether the sample is distinct.

We show that $y \mapsto \mathcal{R}[\hat{H}_{y,N}^{-1}(G(\cdot); \omega')]$ is continuous for each $\omega' \in \Omega'$. Let $y_k \rightarrow y$ and fix $\omega' \in \Omega'$. Using the fact that ν is the uniform distribution and $P \circ G^{-1} = \nu$, we have

$$\begin{aligned} \int_{\Omega} |\hat{H}_{y_k,N}^{-1}(G(\omega); \omega') - \hat{H}_{y,N}^{-1}(G(\omega); \omega')|^p dP(\omega) &= \int_0^1 |\hat{H}_{y_k,N}^{-1}(q; \omega') - \hat{H}_{y,N}^{-1}(q; \omega')|^p d\nu(q) \\ &= \frac{1}{N} \sum_{i=1}^N |f(y_k, \xi^i(\omega')) - f(y, \xi^i(\omega'))|^p. \end{aligned}$$

Since f is a Carathéodory function and $p \in [1, \infty)$, it follows that $\hat{H}_{y_k,N}^{-1}(G(\cdot); \omega') \rightarrow \hat{H}_{y,N}^{-1}(G(\cdot); \omega')$ in $L^p(\Omega, \mathcal{F}, P)$. Combined with the continuity of \mathcal{R} , we have $\mathcal{R}[\hat{H}_{y_k,N}^{-1}(G(\cdot); \omega')] \rightarrow \mathcal{R}[\hat{H}_{y,N}^{-1}(G(\cdot); \omega')]$ as $k \rightarrow \infty$. Consequently, $y \mapsto \mathcal{R}[\hat{H}_{y,N}^{-1}(G(\cdot); \omega')]$ is continuous for each $\omega' \in \Omega'$.

For each fixed $y \in Y_0$, the function $\omega' \mapsto \hat{H}_{y,N}^{-1}(G(\cdot); \omega') \in L^p(\Omega, \mathcal{F}, P)$ is measurable because it is the composition of a piecewise constant and measurable functions.

Combining these arguments, we find that $(y, \omega') \mapsto \mathcal{R}[\hat{H}_{y,N}^{-1}(G(\cdot); \omega')]$ is a Carathéodory function. \square

Corollary 14. *Under the hypotheses of Lemma 13, (a) \mathcal{S} is nonempty and closed, (b) $\hat{\mathcal{S}}_N^r$ has nonempty, closed images for each $r \in [0, \infty)$, and (c) \hat{m}_N^* and $\hat{\mathcal{S}}_N^r$ are measurable for each $r \in [0, \infty)$.*

Proof. (a) Since the set Z_{ad} is nonempty, closed, convex, and bounded, Lemma 12 when combined with the direct method of the calculus of variations ensures the assertions.

(b) Using the properties of Z_{ad} listed in part (a), Lemma 13 when combined with the direct method of the calculus of variations and the complete continuity of \mathbf{B} ensures the assertions.

(c) Since \mathbf{B} is completely continuous and Z is a Banach space, \mathbf{B} is continuous. Lemma 13, the continuity of \mathbf{B} , and Theorem 8.2.11 in [5] imply the measurability assertions. \square

Proof of Theorem 11. To establish the consistency statements, we verify the hypotheses of Corollaries 8 and 22. Corollary 14 ensures that \mathcal{S} is nonempty. Hence $\text{dist}(\cdot, \mathcal{S})$ is (Lipschitz) continuous [1, Thm. 3.16]. Corollary 14 implies that \hat{m}_N^* is measurable and that $\hat{\mathcal{S}}_N^{r,N}$ is measurable with closed, nonempty images. Combined with Theorem 8.2.11 in [5], it follows that $\mathbb{D}(\hat{\mathcal{S}}_N^{r,N}, \mathcal{S})$ is measurable.

Corollary 22 ensures that for almost all $\omega' \in \Omega'$, $Z_{\text{ad}} \ni z \mapsto \mathcal{R}[\hat{H}_{\mathbf{B}z,N}^{-1}(G(\cdot); \omega')]$ Mosco-epiconverges to $Z_{\text{ad}} \ni z \mapsto \mathcal{R}[F(\mathbf{B}z)]$ as $N \rightarrow \infty$. We have $\hat{\mathcal{S}}_N^{r,N} \subset Z_{\text{ad}}$. Moreover, $\hat{\mathcal{S}}_N^{r,N}$ and $\mathcal{S} \subset Z_{\text{ad}}$ are nonempty, and \wp is continuous and convex. Now, for almost all $\omega' \in \Omega'$, Corollary 8 ensures that $\hat{m}_N^*(\omega') \rightarrow m^*$ as $N \rightarrow \infty$. Hence w.p. 1, $\hat{m}_N^* \rightarrow m^*$ as $N \rightarrow \infty$. If furthermore \wp is an R-function, then for almost all $\omega' \in \Omega'$, Corollary 8 ensures $\mathbb{D}(\hat{\mathcal{S}}_N^{r,N}(\omega'), \mathcal{S}) \rightarrow 0$ as $N \rightarrow \infty$. Hence w.p. 1, $\mathbb{D}(\hat{\mathcal{S}}_N^{r,N}, \mathcal{S}) \rightarrow 0$ as $N \rightarrow \infty$. Since \hat{m}_N^* and $\mathbb{D}(\hat{\mathcal{S}}_N^{r,N}, \mathcal{S})$ are measurable, we obtain the almost sure convergence statements. \square

4 Applications

We conclude with the application of our main result, Theorem 11, to several problem classes.

4.1 Consistency of Epi-Regularized and Smoothed Empirical Approximations

Using Theorem 11, we demonstrate the consistency of solutions to epi-regularized and smoothed risk-averse programs using the average value-at-risk. These types of risk measures are popular in numerical approaches, see [36, 38, 39, 6, 72, 15]. For $\beta \in [0, 1)$, the average value-at-risk $\text{AVaR}_\beta : L^1(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ is defined by

$$\text{AVaR}_\beta[X] = \inf_{t \in \mathbb{R}} \left\{ t + \frac{1}{1-\beta} \mathbb{E}[(X - t)^+] \right\},$$

where $(x)^+ := \max\{0, x\}$ for $x \in \mathbb{R}$. Throughout the section, m^* and \mathcal{S} denotes the optimal value and 0-solution set of (8), respectively, with the risk measure $\mathcal{R} = \text{AVaR}_\beta$. Moreover, we denote by m_N^* the optimal value and by $\hat{\mathcal{S}}_N^r$ the r -solution set ($r \geq 0$) of the problem's empirical approximation. The average value-at-risk AVaR_β is a law invariant risk measure [67].

Epi-regularization of risk measures has been proposed and analyzed in [38]. We apply the epi-regularization to the average value-at-risk. As in Example 2 in [38], we consider AVaR_β as defined on $L^2(\Omega, \mathcal{F}, P)$ throughout the remainder of this section. We define $\Phi : L^2(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ by

$$\Phi[X] := (1/2)\mathbb{E}[X^2] + \mathbb{E}[X].$$

For $\varepsilon > 0$, the epi-regularization $\text{AVaR}_\beta^\varepsilon : L^2(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ of AVaR_β is given by

$$\text{AVaR}_\beta^\varepsilon[X] := \inf_{Y \in L^2(\Omega, \mathcal{F}, P)} \left\{ \text{AVaR}_\beta[X - Y] + \varepsilon \Phi[\varepsilon^{-1}Y] \right\}. \quad (11)$$

The risk functional $\text{AVaR}_\beta^\varepsilon$ can be shown to be law invariant. See Appendix B.

For $\varepsilon > 0$, we consider the epi-regularized empirical average value-at-risk optimization problem

$$\min_{z \in \mathcal{Z}_{\text{ad}}} \left\{ \text{AVaR}_\beta^\varepsilon[\hat{H}_{\mathbf{B}z, N}] + \varphi(z) \right\}.$$

We let $\hat{m}_{\text{epi}, N}^\varepsilon$ be its optimal value and $\hat{\mathcal{S}}_{\text{epi}, N}^\varepsilon$ be its 0-solution set. Note that for fixed $\varepsilon > 0$, our main result, Theorem 11, already provides an asymptotic consistency result. However, in numerical procedures, the ε -parameter is typically driven to zero. Therefore, we prove a stronger statement here.

Proposition 15. *Let Assumption 10 hold with $p = 2$. Suppose further that (Ω, \mathcal{F}, P) is nonatomic and complete. Let $(\varepsilon_N) \subset (0, \infty)$ with $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$. Then $\hat{m}_{\text{epi}, N}^{\varepsilon_N} \rightarrow m^*$ w.p. 1 as $N \rightarrow \infty$. If furthermore φ is an R -function, then $\mathbb{D}(\hat{\mathcal{S}}_{\text{epi}, N}^{\varepsilon_N}, \mathcal{S}) \rightarrow 0$ w.p. 1 as $N \rightarrow \infty$.*

The proof of Proposition 15 is based on the following result.

Lemma 16. *Fix $\varepsilon > 0$. The functional $\text{AVaR}_\beta^\varepsilon : L^2(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ is a law invariant, convex risk measure. For all $X \in L^2(\Omega, \mathcal{F}, P)$, it holds that*

$$\text{AVaR}_\beta[X] - \frac{\varepsilon\beta}{2(1-\beta)} \leq \text{AVaR}_\beta^\varepsilon[X] \leq \text{AVaR}_\beta[X].$$

Proof. The functional $\text{AVaR}_\beta^\varepsilon$ is a convex risk measure [38, pp. 776 and 778–779]. By the arguments in Appendix B, it is law invariant. Since $\Phi[0] = 0$, the second estimate is implied by Proposition 1 in [38]. Fix $X \in L^2(\Omega, \mathcal{F}, P)$. Since AVaR_β is subdifferentiable [68, p. 243], Proposition 2 in [38] yields for all subgradients $\vartheta \in \partial \text{AVaR}_\beta[X]$ (see, e.g., p. 480 in [68] for definitions of subgradients and subdifferentials),

$$\text{AVaR}_\beta^\varepsilon[X] \geq \text{AVaR}_\beta[X] - \varepsilon \Phi^*[\vartheta].$$

Here Φ^* is the Fenchel conjugate to Φ ; see, e.g., p. 232 in [68] for a definition. Let $\vartheta \in \partial \text{AVaR}_\beta[X]$ be arbitrary. We have $0 \leq \vartheta \leq 1/(1-\beta)$ w.p. 1, $\mathbb{E}[\vartheta] = 1$ [68, p. 243] and $\Phi^*[\vartheta] = (1/2)\mathbb{E}[(\vartheta - 1)^2]$; see Remark 5 in [38]. Hence

$$\Phi^*[\vartheta] = (1/2)\mathbb{E}[\vartheta^2] - \mathbb{E}[\vartheta] + (1/2) = (1/2)\mathbb{E}[\vartheta^2] - (1/2) \leq \frac{1}{2} \frac{1 - (1-\beta)}{1-\beta} = \frac{1}{2} \frac{\beta}{1-\beta}.$$

□

Proof of Proposition 15. Following the proof of Corollary 14 and using the fact that $\text{AVaR}_\beta^{\varepsilon_N}$ is a law invariant, convex risk measure (see Lemma 16), we find that $\hat{m}_{\text{epi},N}^{\varepsilon_N}$ and $\mathcal{S}_{\text{epi},N}^{\varepsilon_N}$ are measurable. Lemma 16 ensures that $\hat{m}_N^* - \frac{\varepsilon_N \beta}{2(1-\beta)} \leq \hat{m}_{\text{epi},N}^{\varepsilon_N} \leq \hat{m}_N^*$. Applying Theorem 11 with $\mathcal{R} = \text{AVaR}_\beta$ yields $\hat{m}_N^* \rightarrow m^*$ w.p. 1 as $N \rightarrow \infty$. Combined with $\varepsilon_N \rightarrow 0$, we find that $\hat{m}_{\text{epi},N}^{\varepsilon_N} \rightarrow m^*$ w.p. 1 as $N \rightarrow \infty$.

If $z_N^{\varepsilon_N} \in \hat{\mathcal{S}}_{\text{epi},N}^{\varepsilon_N}$, then Lemma 16 ensures that $z_N^{\varepsilon_N} \in \hat{\mathcal{S}}_N^{r_N}$, where $r_N := \frac{\varepsilon_N \beta}{2(1-\beta)}$. Hence $\hat{\mathcal{S}}_{\text{epi},N}^{\varepsilon_N} \subset \hat{\mathcal{S}}_N^{r_N}$, yielding $\mathbb{D}(\hat{\mathcal{S}}_{\text{epi},N}^{\varepsilon_N}, \mathcal{S}) \leq \mathbb{D}(\hat{\mathcal{S}}_N^{r_N}, \mathcal{S})$. Applying Theorem 11 with $\mathcal{R} = \text{AVaR}_\beta$ yields the second assertion. □

Next, we establish the consistency of solutions to smoothed average value-at-risk problems using a smoothing function for $(\cdot)^+$. For brevity, we focus on a particular smoothing function for the plus function $(\cdot)^+$. For $\varepsilon > 0$, we define the smoothed plus function $(\cdot)_\varepsilon^+ : \mathbb{R} \rightarrow \mathbb{R}$ by

$$(x)_\varepsilon^+ := \varepsilon \ln(1 + \exp(x/\varepsilon)).$$

Using $(\cdot)_\varepsilon^+$, we define the smoothed average value-at-risk $\sigma_\beta^\varepsilon : L^2(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ by

$$\sigma_\beta^\varepsilon[X] := \inf_{t \in \mathbb{R}} \left\{ t + \frac{1}{1-\beta} \mathbb{E}[(X - t)_\varepsilon^+] \right\}. \quad (12)$$

This version of the smoothed average value-at-risk has been used in [72] for stochastic stellarator coil design and in [6] for adaptive sampling techniques for risk-averse optimization. See the Appendix B for a short proof of its law invariance.

For $\varepsilon > 0$, we consider the smoothed empirical average value-at-risk optimization problem

$$\min_{z \in Z_{\text{ad}}} \left\{ \sigma_\beta^\varepsilon[\hat{H}_{\mathbf{B}z, N}] + \wp(z) \right\},$$

We let $\hat{m}_{\text{s},N}^\varepsilon$ be its optimal value and $\hat{\mathcal{S}}_{\text{s},N}^\varepsilon$ be its 0-solution set.

Proposition 17. *Let Assumption 10 hold with $p = 2$. Suppose further that (Ω, \mathcal{F}, P) is nonatomic and complete. Let $(\varepsilon_N) \subset (0, \infty)$ with $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$. Then $\hat{m}_{\text{s},N}^{\varepsilon_N} \rightarrow m^*$ w.p. 1 as $N \rightarrow \infty$. If furthermore \wp is an R-function, then $\mathbb{D}(\hat{\mathcal{S}}_{\text{s},N}^{\varepsilon_N}, \mathcal{S}) \rightarrow 0$ w.p. 1 as $N \rightarrow \infty$.*

Proposition 17 is established using Lemma 18.

Lemma 18. *Fix $\varepsilon > 0$. The functional $\sigma_\beta^\varepsilon : L^2(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ is a law invariant, convex risk measure. For all $X \in L^2(\Omega, \mathcal{F}, P)$, it holds that*

$$\text{AVaR}_\beta[X] \leq \sigma_\beta^\varepsilon[X] \leq \text{AVaR}_\beta[X] + \ln(2)\varepsilon/(1-\beta).$$

Proof. The smoothed average value-at-risk σ_β^ε is a convex risk measure [36, Props. 4.4–4.6]. By the arguments in Appendix B, it is law invariant. For $x \in \mathbb{R}$, we have $(x)^+ \leq (x)_\varepsilon^+ \leq (x)^+ + \varepsilon \ln(2)$, yielding the error bounds. □

Proof of Proposition 17. The proof is similar to that of Proposition 15. Following the proof of Corollary 14 and using the fact that $\sigma_\beta^{\varepsilon_N}$ is a law invariant, convex risk measure (see Lemma 18), we find that $\hat{m}_{s,N}^{\varepsilon_N}$ and $\hat{\mathcal{S}}_{s,N}^{\varepsilon_N}$ are measurable. Lemma 16 ensures that $\hat{m}_N^* \leq \hat{m}_{s,N}^{\varepsilon_N} \leq \hat{m}_N^* + \ln(2)\varepsilon_N/(1-\beta)$. Applying Theorem 11 with $\mathcal{R} = \text{AVaR}_\beta$ yields $\hat{m}_N^* \rightarrow m^*$ w.p. 1 as $N \rightarrow \infty$. Combined with $\varepsilon_N \rightarrow 0$, we find that $\hat{m}_{s,N}^{\varepsilon_N} \rightarrow m^*$ w.p. 1 as $N \rightarrow \infty$.

If $z_N^{\varepsilon_N} \in \hat{\mathcal{S}}_{s,N}^{\varepsilon_N}$, then Lemma 16 ensures that $z_N^{\varepsilon_N} \in \hat{\mathcal{S}}_N^{r_N}$, where $r_N := \ln(2)\varepsilon_N/(1-\beta)$. Hence $\hat{\mathcal{S}}_{s,N}^{\varepsilon_N} \subset \hat{\mathcal{S}}_N^{r_N}$, yielding $\mathbb{D}(\hat{\mathcal{S}}_{s,N}^{\varepsilon_N}, \mathcal{S}) \leq \mathbb{D}(\hat{\mathcal{S}}_N^{r_N}, \mathcal{S})$. Applying Theorem 11 with $\mathcal{R} = \text{AVaR}_\beta$ yields the second assertion. \square

4.2 Risk-Averse Semilinear PDE-Constrained Optimization

Our consistency result, Theorem 11, is applicable to risk-averse semilinear PDE-constrained optimization as we demonstrate in this section. Following [39] (see also [25, 26]), we consider

$$\min_{z \in Z_{\text{ad}}} (1/2)\mathcal{R}[\|(1 - \iota S(z))^+\|_{L^2(D)}^2] + (\alpha/2)\|z\|_{L^2(D)}^2, \quad (13)$$

where $\alpha > 0$, $\iota : H^1(D) \rightarrow L^2(D)$ is the embedding operator of the compact embedding $H^1(D) \hookrightarrow L^2(D)$, $Z_{\text{ad}} := \{z \in L^2(D) : \mathbf{l}(x) \leq z(x) \leq \mathbf{u}(x) \text{ for a.e. } x \in D\}$ with $\mathbf{l}, \mathbf{u} \in L^2(D)$ and $\mathbf{l}(x) \leq \mathbf{u}(x)$ for a.e. $x \in D$, and for each $(z, \xi) \in L^2(D) \times \Xi$, $S(z)(\xi) \in H^1(D)$ is the solution to:

$$\text{find } u \in H^1(D): \quad \mathbf{A}(u, \xi) = \mathbf{B}_1(\xi)\iota^*z + \mathbf{b}(\xi), \quad (14)$$

where ι^* is the adjoint operator to ι , $\mathbf{A} : H^1(D) \times \Xi \rightarrow H^1(D)^*$, $\mathbf{B}_1 : \Xi \rightarrow \mathcal{L}(H^1(D)^*, H^1(D)^*)$, and $\mathbf{b} : \Xi \rightarrow H^1(D)^*$ are defined by

$$\begin{aligned} \langle \mathbf{A}(u, \xi), v \rangle_{H^1(D)^*, H^1(D)} &:= \int_D a(\xi)(x) [\nabla u(x)^T \nabla v(x) + u(x)v(x)] dx + \int_D u(x)^3 v(x) dx, \\ \langle \mathbf{B}_1(\xi)y, v \rangle_{H^1(D)^*, H^1(D)} &:= \int_D [B(\xi)y](x)v(x) dx, \quad \langle \mathbf{b}(\xi), v \rangle_{H^1(D)^*, H^1(D)} := \int_D b(\xi)(x)v(x) dx. \end{aligned}$$

Here, $b : \Xi \rightarrow L^2(D)$ is essentially bounded, $a : \Xi \rightarrow C^0(\bar{D})$ is measurable and there exist constants $\kappa_{\min}, \kappa_{\max} > 0$ such that $\kappa_{\min} \leq a(\xi)(x) \leq \kappa_{\max}$ for all $(\xi, x) \in \Xi \times \bar{D}$. It remains to define $B(\xi) : H^1(D)^* \rightarrow H^1(D)$. Fix $(y, \xi) \in H^1(D)^* \times \Xi$. We define $B(\xi)y \in H^1(D)$ as the solution to: find $w \in H^1(D)$ such that

$$\int_D [r(\xi)\nabla w(x)^T \nabla v(x) + w(x)v(x)] dx = \langle y, v \rangle_{H^1(D)^*, H^1(D)} \quad \text{for all } v \in H^1(D),$$

where $r : \Xi \rightarrow (0, \infty)$ is random variable such that there exist $r_{\min}, r_{\max} > 0$ with $r_{\min} \leq r(\xi) \leq r_{\max}$ for all $\xi \in \Xi$. Since $\iota u = u$ for all $u \in H^1(D)$, we have $\langle \iota^*z, v \rangle_{H^1(D)^*, H^1(D)} = \langle z, v \rangle_{L^2(D)}$ for all $z \in L^2(D)$ and $v \in H^1(D)$ [12, p. 21].

We express (13) in the form given in (8) and verify Assumption 10. For each $(y, \xi) \in H^1(D)^* \times \Xi$, we consider the auxiliary random operator equation:

$$\text{find } u \in H^1(D): \quad \mathbf{A}(u, \xi) = \mathbf{B}_1(\xi)y + \mathbf{b}(\xi). \quad (15)$$

Lemma 19. *Under the above hypotheses, for each $(y, \xi) \in H^1(D)^* \times \Xi$, the operator equation (15) has a unique solution $\tilde{S}(y)(\xi)$, $\tilde{S}(y) \in L^q(\Xi, \mathcal{A}, \mathbb{P}; H^1(D))$ for each $q \in [1, \infty]$ and $y \in H^1(D)^*$, $(y, \xi) \mapsto \tilde{S}(z)(\xi)$ is a Carathéodory mapping, and $\tilde{S} : H^1(D)^* \rightarrow L^q(\Xi, \mathcal{A}, \mathbb{P}; H^1(D))$ is Lipschitz continuous for each $q \in [1, \infty]$.*

Let U be a reflexive Banach space. We recall that an operator $A : U \rightarrow U^*$ is κ -strongly monotone if there exists $\kappa > 0$ such that

$$\langle A(u_2) - A(u_1), u_2 - u_1 \rangle_{U^*, U} \geq \kappa \|u_2 - u_1\|_U^2 \quad \text{for all } u_1, u_2 \in U.$$

Proof of Lemma 19. For each $\xi \in \Xi$, $\mathbf{A}(\cdot, \xi)$ is κ_{\min} -strongly monotone and it holds that

$$\|\mathbf{B}_1(\xi)\|_{\mathcal{L}(H^1(D)^*, H^1(D)^*)} \leq 1/\min\{r_{\min}, 1\};$$

cf. [39, p. 13]. The existence, uniqueness and the stability estimate

$$\|\tilde{S}(y)(\xi)\|_{H^1(D)} \leq (1/\kappa_{\min})\|\mathbf{B}_1(\xi)y\|_{H^1(D)^*} + (1/\kappa_{\min})\|\mathbf{b}(\xi)\|_{H^1(D)^*}$$

are a consequence of the Minty–Browder theorem [73, Thm. A.26], for example. Using Filippov’s theorem [5, Thm. 8.2.10], we can show that $\tilde{S}(y)$ is measurable. Combined with the stability estimate and Hölder’s inequality, we conclude that $\tilde{S}(y) \in L^q(\Xi, \mathcal{A}, \mathbb{P}; H^1(D))$ for each $q \in [1, \infty]$ and $y \in H^1(D)^*$. Since for all $y_1, y_2 \in H^1(D)^*$ and $\xi \in \Xi$, we have (cf. [39, eq. (3.7)])

$$\|\tilde{S}(y_2)(\xi) - \tilde{S}(y_1)(\xi)\|_U \leq (1/\kappa_{\min})\|\mathbf{B}_1(\xi)[y_2 - y_1]\|_{H^1(D)^*},$$

the mapping $(y, \xi) \mapsto \tilde{S}(y)(\xi)$ is a Carathéodory mapping, and $\tilde{S} : H^1(D)^* \rightarrow L^q(\Xi, \mathcal{A}, \mathbb{P}; H^1(D))$ is Lipschitz continuous for all $q \in [1, \infty]$. \square

The function \wp defined by $\wp(z) := (\alpha/2)\|z\|_{L^2(D)}^2$ is an R-function according to Lemma 1, as $\alpha > 0$ and $L^2(D)$ is a Hilbert space and hence has the Radon–Riesz property [12, Prop. 2.35]. The operator $\mathbf{B} := \iota^*$ is linear and completely continuous because ι is a compact operator by the Sobolev embedding theorem. We define $f : H^1(D)^* \times \Xi \rightarrow [0, \infty)$ by $f(y, \xi) := (1/2)\|(1 - \iota\tilde{S}(y)(\xi))^+\|_{L^2(D)}^2$. The mapping $\mathcal{J} : L^q(\Xi, \mathcal{A}, \mathbb{P}; H^1(D)) \rightarrow L^{q/2}(\Xi, \mathcal{A}, \mathbb{P})$ given by $\mathcal{J}(u) := (1/2)\|(1 - \iota u)^+\|_{L^2(D)}^2$ is continuous for $q \in [2, \infty)$ [38, Proposition 5]. Lemma 19 ensures that $\mathcal{J} \circ \tilde{S} : H^1(D)^* \rightarrow L^{q/2}(\Xi, \mathcal{A}, \mathbb{P})$ is well-defined and continuous for $q \in [2, \infty)$, yielding the continuity of F with $p = q/2$. Having verified Assumption 10 for $p \in [2, \infty)$, we can apply Theorem 11 to study the consistency of empirical approximations of (13).

4.3 Risk-Averse Optimization with Variational Inequalities

We consider a risk-averse optimization problem governed by an elliptic variational inequality with random inputs. Our presentation is inspired by that in [28]. We consider

$$\min_{z \in Z_{\text{ad}}} (1/2)\mathcal{R}[\|\iota S(z) - u_d\|_{L^2(D)}^2] + (\alpha/2)\|z\|_{L^2(D)}^2, \quad (16)$$

where $\alpha > 0$, $u_d \in L^2(D)$, $\iota : H_0^1(D) \rightarrow L^2(D)$ is the embedding operator of the compact embedding $H_0^1(D) \hookrightarrow L^2(D)$, and Z_{ad} is as in Section 4.2. For each $(z, \xi) \in L^2(D) \times \Xi$, $S(z)(\xi) \in H_0^1(D)$ is the solution to the parameterized elliptic variational inequality:

$$\text{find } u \in K_\psi : \quad \langle A(\xi)u - \iota^*z, v - u \rangle_{H^{-1}(D), H_0^1(D)} \geq 0 \quad \text{for all } v \in K_\psi, \quad (17)$$

where ι^* is the adjoint operator to ι , $H^{-1}(D) := H_0^1(D)^*$, $A : \Xi \rightarrow \mathcal{L}(H_0^1(D), H^{-1}(D))$ is a parameterized elliptic operator, and $K_\psi := \{u \in H_0^1(D) : u(x) \geq \psi(x) \text{ for a.e. } x \in D\}$ with $\psi \in$

$H^1(D)$ and $\psi_{\partial D} \leq 0$ is the obstacle. The set K_ψ is nonempty [71, p. 129]. For $(y, \xi) \in H^{-1}(D) \times \Xi$, we also consider the auxiliary parameterized elliptic variational inequality:

$$\text{find } u \in K_\psi: \quad \langle A(\xi)u - y, v - u \rangle_{H^{-1}(D), H_0^1(D)} \geq 0 \quad \text{for all } v \in K_\psi. \quad (18)$$

If $\tilde{S}(y)(\xi)$ with $y = \iota^*z$ is a solution to (18), then it is a solution to (17).

We assume that $A : \Xi \rightarrow \mathcal{L}(H_0^1(D), H^{-1}(D))$ is uniformly measurable, that is, there exists a sequence $A_k : \Xi \rightarrow \mathcal{L}(H_0^1(D), H^{-1}(D))$ of simple mappings such that $A_k(\xi) \rightarrow A(\xi)$ in $\mathcal{L}(H_0^1(D), H^{-1}(D))$ as $k \rightarrow \infty$ for each $\xi \in \Xi$. Moreover, we assume that there exist constants $\kappa_{\min}, \kappa_{\max} > 0$ such that for each $\xi \in \Xi$, $A(\xi)$ is κ_{\min} -strongly monotone and $\|A(\xi)\|_{\mathcal{L}(H_0^1(D), H^{-1}(D))} \leq \kappa_{\max}$. Under these conditions, the auxiliary variational inequality (17) has a unique solution $\tilde{S}(y)(\xi)$ for each $(y, \xi) \in H^{-1}(D) \times \Xi$, and $\tilde{S}(\cdot)(\xi)$ is Lipschitz continuous with Lipschitz constant $1/\kappa_{\min}$ for each $\xi \in \Xi$; cf. [28, Thm. 7.3]. Using results established in [27, p. 180], we can show that $\tilde{S}(y) \in L^q(\Xi, \mathcal{A}, \mathbb{P}; H_0^1(D))$ for all $q \in [1, \infty]$ and $y \in H^{-1}(D)$. Combined with the Lipschitz continuity, we find that $\tilde{S} : H^{-1}(D) \rightarrow L^q(\Xi, \mathcal{A}, \mathbb{P}; H_0^1(D))$ is continuous for each $q \in [1, \infty]$.

We express (16) in the form given in (8) and verify Assumption 10. The function \wp defined by $\wp(z) := (\alpha/2)\|z\|_{L^2(D)}^2$ is an R-function; see Section 4.2. The operator $\mathbf{B} := \iota^*$ is linear and completely continuous because ι is a compact operator. We define $f : H^{-1}(D) \times \Xi \rightarrow [0, \infty)$ by $f(y, \xi) := (1/2)\|\iota\tilde{S}(y)(\xi) - u_d\|_{L^2(D)}^2$. The mapping $\mathcal{J} : L^q(\Xi, \mathcal{A}, \mathbb{P}; H_0^1(D)) \rightarrow L^{q/2}(\Xi, \mathcal{A}, \mathbb{P})$ given by $\mathcal{J}(u) := (1/2)\|\iota u - u_d\|_{L^2(D)}^2$ is continuous for $q \in [2, \infty)$; cf. [37, Example 3.2 and Theorem 3.5]. Combined with the continuity of \tilde{S} , we find that $\mathcal{J} \circ \tilde{S} : H^{-1}(D) \rightarrow L^{q/2}(\Xi, \mathcal{A}, \mathbb{P})$ is well-defined and continuous for $q \in [2, \infty)$, yielding the continuity of F with $p = q/2$. Having verified Assumption 10 for $p \in [1, \infty)$, we can apply Theorem 11, which in turn yields the consistency of empirical approximations of (16).

5 Conclusion

We have seen that consistency results, in particular, norm consistency of empirical minimizers for nonconvex, risk-averse stochastic optimization problems involving infinite dimensional decision spaces are in fact available. The central property on which the entire discussion depends is the ability to draw compactness from the structure of the objective function. As the examples illustrate, this is much more the rule rather than the exception. In fact, even in examples such as topology optimization, [8], where the decision variable enters the PDE in a nonlinear fashion, the required use of either filters or other regularization strategies, see e.g. [41, 70], also provides compactness.

There remain many open challenges. These include applications to multistage or dynamic problems, large deviation results for optimal values and solutions, and central limit theorems. In many instances, the known techniques are limited by nonsmoothness of the risk measure \mathcal{R} and the infinite dimensional decision spaces. However, the main result in this text, Theorem 11, is a first major step and an essential tool towards verifying the convergence of numerical optimization methods that make use of empirical approximations. Moreover, for numerical computations, the decision spaces of infinite dimensional risk-averse optimization problems must typically be discretized. Therefore, in a practical setting, these problems have the additional challenge that the numerically computed estimators are generally dependent on both the sample size N and additional spacial discretization parameters. As initial contributions for risk-neutral PDE-constrained problems [30, 45] demonstrate, an infinite dimensional consistency analysis provides an important component in the numerical analysis of these challenging optimization problems.

A Law of Large Numbers for Risk Functionals

We generalize the epigraphical law of large numbers for law invariant risk function established in Theorem 3.1 in [66] to allow for random lower semicontinuous functions defined on complete, separable metric spaces instead of \mathbb{R}^n . The proof of Theorem 3.1 provided in [66] generalizes to this more general setting with only a few notational changes needed. Nevertheless, we verify the liminf-condition of epiconvergence using ideas from the proof of Proposition 7.1 in [60]. The limsup-condition is established as in [66].

Assumption 20. Let (Ω, \mathcal{F}, P) be a nonatomic, complete probability space, and let $(\Theta, \Sigma, \mathbb{M})$ be a complete probability space. Let $\zeta : \Omega \rightarrow \Theta$ be a random element with distribution \mathbb{M} and let ζ^1, ζ^2, \dots defined on a complete probability space $(\Omega', \mathcal{F}', P')$ be independent identically distributed Θ -valued random elements each having the same distribution as that of ζ . Let (V, d_V) be a complete, separable metric space and let $1 \leq p < \infty$.

($\mathfrak{A}1$) The function $\Psi : V \times \Theta \rightarrow \mathbb{R}$ is random lower semicontinuous.

($\mathfrak{A}2$) For each $v \in V$, $\Psi_v(\cdot) := \Psi(v, \cdot) \in L^p(\Theta, \Sigma, \mathbb{M})$.

($\mathfrak{A}3$) For each $\bar{v} \in V$, there exists a neighborhood $\mathcal{V}_{\bar{v}} \subset V$ of \bar{v} and a random variable $h \in L^p(\Theta, \Sigma, \mathbb{M})$ such that $\Psi(v, \cdot) \geq h(\cdot)$ for all $v \in \mathcal{V}_{\bar{v}}$.

Theorem 21 is as Theorem 3.1 in [66] but allows for complete, separable metric spaces V instead of \mathbb{R}^n . Let $\rho : L^p(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ be a law invariant risk measure and let Assumption 20 hold true. Let $v \in V$ and let $\hat{H}_{v,N}(\cdot; \omega')$ be the empirical distribution function of $\Psi(v, \zeta^1(\omega')), \dots, \Psi(v, \zeta^N(\omega'))$. Moreover, let $\hat{H}_{v,N}^{-1}(\cdot; \omega')$ be its quantile function. We define $\hat{\phi}_N : V \times \Omega' \rightarrow \mathbb{R}$ and $\phi : V \rightarrow \mathbb{R}$ by $\hat{\phi}_N(v, \omega') := \rho(\hat{H}_{v,N}^{-1}(G(\cdot); \omega'))$ and $\phi(v) := \rho(\Psi_v(\zeta)) = \rho(\Psi_v(\zeta(\cdot)))$. Here $G : \Omega \rightarrow [0, 1]$ is a random variable with uniform distribution ν as discussed in Section 2.1. We often omit writing the second argument of $\hat{\phi}_N$.

Theorem 21. *If Assumption 20 holds and $\rho : L^p(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ is a law invariant, convex risk measure, then ϕ is lower semicontinuous and finite-valued, and $\hat{\phi}_N$ epiconverges to ϕ w.p. 1 as $N \rightarrow \infty$.*

Before establishing Theorem 21, we formulate a law of large numbers with respect to Mosco-epiconvergence.

Corollary 22. *Let $Y_0 \subset Y$ be a closed subset of a separable Banach space Y and let W_0 be a nonempty, closed, convex subset of a reflexive, separable Banach space W . Let the hypotheses of Theorem 21 hold with $V = Y_0$. Suppose that $\mathbf{B} : W \rightarrow Y$ is linear and completely continuous with $\mathbf{B}(W_0) \subset Y_0$. Then $\hat{\phi}_N \circ \mathbf{B} : W_0 \rightarrow \mathbb{R}$ Mosco-epiconverges to $\phi \circ \mathbf{B} : W_0 \rightarrow \mathbb{R}$ w.p. 1 as $N \rightarrow \infty$.*

Proof. Theorem 21 ensures that $\hat{\phi}_N$ epiconverges to ϕ w.p. 1 as $N \rightarrow \infty$. Since W_0 defines a complete separable metric space, $\mathbf{B}(W_0) \subset Y_0$, and \mathbf{B} is continuous, Theorem 21 further ensures that $\hat{\phi}_N \circ \mathbf{B}$ epiconverges to $\phi \circ \mathbf{B}$ w.p. 1 as $N \rightarrow \infty$. Combined with Proposition 9 and the complete continuity of \mathbf{B} , we conclude that $\hat{\phi}_N \circ \mathbf{B}$ Mosco-epiconverges to $\phi \circ \mathbf{B}$ w.p. 1 as $N \rightarrow \infty$. \square

As already mentioned, the proof of Theorem 21 presented in [66, Thm. 3.1] for $V = \mathbb{R}^n$ can be generalized to the above setting without much effort. A key result for establishing Theorem 21 is Theorem 23. To formulate Theorem 23, let $X \in L^p(\Omega, \mathcal{F}, P)$ be a random variable and X_1, X_2, \dots defined on a complete probability space $(\Omega', \mathcal{F}', P')$ be independent identically distributed real-valued random variables each having the same distribution as that of X . Moreover, let $\hat{H}_N(\cdot; \omega')$

be the empirical distribution function of the sample $X_1(\omega'), \dots, X_N(\omega')$, and $\hat{H}_N^{-1}(\cdot; \omega')$ be its quantile function.

Theorem 23 (see [66, Thm. 2.1] and [68, Thm. 9.65]). *If (Ω, \mathcal{F}, P) is complete and nonatomic, $1 \leq p < \infty$, and $\rho : L^p(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ is a law invariant, convex risk measure, then $\rho(\hat{H}_N)$ converges to $\rho(X)$ w.p. 1 as $N \rightarrow \infty$.*

Proof. We present a proof somewhat different from that in [66]. Fix $\omega' \in \Omega'$. Using a change of variables and the fact that $G : \Omega \rightarrow [0, 1]$ has uniform distribution ν , we obtain

$$\begin{aligned} \int_{\Omega} |\hat{H}_N^{-1}(G(\omega); \omega') - H_X^{-1}(G(\omega))|^p dP(\omega) &= \int_0^1 |\hat{H}_N^{-1}(q; \omega') - H_X^{-1}(q)|^p dP \circ G^{-1}(q) \\ &= \int_0^1 |\hat{H}_N^{-1}(q; \omega') - H_X^{-1}(q)|^p d\nu(q). \end{aligned}$$

Since $1 \leq p < \infty$, $X \in L^p(\Omega, \mathcal{F}, P)$ and X_1, X_2, \dots defined on $(\Omega', \mathcal{F}', P')$ are independent identically distributed each with the same distribution as that of X , the latter integral converges P' -almost surely to zero as $N \rightarrow \infty$; see [54, Cor. on p. 48], [52, Cor. 3], [53, Cor. 3 on p. 666]. Since $\rho : L^p(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ is a real-valued convex risk measure, it is continuous [61, Cor. 3.1]. We obtain for almost every $\omega' \in \Omega'$, $\rho(\hat{H}_N^{-1}(G(\cdot); \omega')) \rightarrow \rho(H_X^{-1}(G(\cdot)))$ as $N \rightarrow \infty$. Combined with $\rho(X) = \rho(H_X^{-1}(G(\cdot)))$ and $\rho(\hat{H}_N) = \rho(\hat{H}_N^{-1}(G(\cdot)))$, we obtain the assertion. \square

Proof of Theorem 21. The fact that ϕ is finite-valued and lower semicontinuous can be established as in the proof of Theorem 3.1 in [66]. To establish the epiconvergence, we make use of the constructions made in the proof of Proposition 7.1 in [60]. Proposition 7.1 in [60] establishes epiconvergence in case that $\rho(\cdot) = \mathbb{E}[\cdot]$, but without assuming (Ω, \mathcal{F}, P) be nonatomic. Let $\mathcal{E} \subset V$ be a countable dense subset of V and Q_+ be the nonnegative rational numbers. For $v \in V$ and $r \in [0, \infty)$, we define $\pi_{v,r}$ on Θ by

$$\pi_{v,r}(\theta) := \inf_{w \in B(v,r)} \Psi(w, \theta) \quad \text{if } r > 0 \quad \text{and} \quad \pi_{v,0}(\theta) := \Psi(v, \theta) \quad \text{if } r = 0,$$

where $B(v, r) := \{w \in V : d_V(w, v) < r\}$. Theorem 3.4 in [35], $(\mathfrak{R}1)$, and $(\mathfrak{R}2)$ ensure that $\pi_{v,r}$ is an extended real-valued random variable for each $v \in V$ and $r \geq 0$. Combined with $(\mathfrak{R}3)$, we find that for every $v \in \mathcal{E}$, there exists a neighborhood $\mathcal{V}_v \subset V$ of v and $r_v \in (0, \infty)$ such that

$$B(v, r_v) \subset \mathcal{V}_v \quad \text{and} \quad \pi_{v,r}(\cdot) \in L^p(\Theta, \Sigma, \mathbb{M}) \quad \text{for all } r \in [0, r_v] \cap Q_+.$$

Let $\tilde{H}_{v,r,N}(\cdot; \omega')$ be the empirical distribution function of $\pi_{v,r}(\zeta^i(\omega'))$, $i = 1, \dots, N$, and let $\tilde{H}_{v,r,N}^{-1}(\cdot; \omega')$ be its quantile function. For every $v \in \mathcal{E}$ and $r \in [0, r_v] \cap Q_+$, Theorem 23 ensures that $\rho(\tilde{H}_{v,r,N}^{-1}(G; \cdot)) \rightarrow \rho(\pi_{v,r}(\zeta))$ w.p. 1 as $N \rightarrow \infty$. Since $\{(v, r) : r \in [0, r_v] \cap Q_+, v \in \mathcal{E}\}$ is countable, there exists $\Omega'_0 \subset \Omega'$ with $\Omega'_0 \in \mathcal{F}'$ and $P'(\Omega'_0) = 1$ such that

$$\rho(\tilde{H}_{v,r,N}^{-1}(G; \omega')) \rightarrow \rho(\pi_{v,r}(\zeta)) \quad \text{as } N \rightarrow \infty \quad \text{for all } \omega' \in \Omega'_0 \quad \text{and} \quad r \in [0, r_v] \cap Q_+, v \in \mathcal{E}.$$

Now, we verify the liminf-condition of epiconvergence. Fix $v \in V$ and fix $v_N \rightarrow v$ as $N \rightarrow \infty$. There exist $z_\ell \in \mathcal{E}$ with $z_\ell \rightarrow v$ as $\ell \rightarrow \infty$, $r_\ell \in (0, r_v] \cap Q_+$ with $r_\ell \rightarrow 0$, and for each $\ell \in \mathbb{N}$, there exists $\bar{N}(\ell) \in \mathbb{N}$ such that

$$v \in B(z_{\ell+1}, r_{\ell+1}) \subset B(z_\ell, r_\ell), \quad \text{and} \quad v_N \in B(z_\ell, r_\ell) \quad \text{for all } N \geq \bar{N}(\ell).$$

Fix $\ell \in \mathbb{N}$. For all $N \geq \bar{N}(\ell)$ and $\omega' \in \Omega'_0$, Theorem 6.50 in [68] when combined with the fact that ρ is law invariant and monotone ensures

$$\hat{\phi}_N(v_N, \omega') = \rho(\hat{H}_{v_N, N}^{-1}(G; \omega')) \geq \rho(\tilde{H}_{z_\ell, r_\ell, N}^{-1}(G; \omega')). \quad (19)$$

Moreover, for all $\omega' \in \Omega'_0$,

$$\rho(\tilde{H}_{z_\ell, r_\ell, N}^{-1}(G; \omega')) \rightarrow \rho(\pi_{z_\ell, r_\ell}(\zeta)) \quad \text{as } N \rightarrow \infty. \quad (20)$$

Since $v \in B(z_{\ell+1}, r_{\ell+1}) \subset B(z_\ell, r_\ell)$, we have $\pi_{z_\ell, r_\ell} \leq \pi_{z_{\ell+1}, r_{\ell+1}} \leq \pi_{v, 0}$. For all $\ell \in \mathbb{N}$ and $\theta \in \Theta$, the lower semicontinuity of $\Psi(\cdot, \theta)$ (see $(\mathfrak{A}1)$) ensures $\pi_{z_\ell, r_\ell}(\theta) \nearrow \pi_{v, 0}(\theta) = \Psi(v, \theta)$ as $\ell \rightarrow \infty$ [35, p. 432]. Thus $\pi_{v, 0} - \pi_{z_1, r_1} \geq \pi_{v, 0} - \pi_{z_{\ell+1}, r_{\ell+1}} \geq 0$ for all $\ell \in \mathbb{N}$. Consequently, $|\pi_{v, 0} - \pi_{z_1, r_1}|^p \geq |\pi_{v, 0} - \pi_{z_{\ell+1}, r_{\ell+1}}|^p$. Since $\pi_{z_1, r_1}(\zeta), \pi_{v, 0}(\zeta) \in L^p(\Omega, \mathcal{F}, P)$, the dominated convergence theorem implies $\pi_{z_\ell, r_\ell}(\zeta) \rightarrow \pi_{v, 0}(\zeta)$ as $\ell \rightarrow \infty$ in $L^p(\Omega, \mathcal{F}, P)$. Using the fact that the risk measure ρ is real-valued and convex, it follows that ρ is continuous [61, Cor. 3.1] and monotone. Consequently, $\rho(\pi_{z_\ell, r_\ell}(\zeta)) \nearrow \rho(\pi_{v, 0}(\zeta)) = \phi(v)$ as $\ell \rightarrow \infty$. Combined with (19) and (20), we find that for all $\omega' \in \Omega'_0$,

$$\liminf_{N \rightarrow \infty} \hat{\phi}_N(v_N, \omega') \geq \phi(v).$$

Now, we verify the limsup-condition of epiconvergence using the arguments in [66]. Since ϕ is defined on a separable metric space, finite-valued and lower semicontinuous, there exists a countable set $\mathcal{D} \subset V$ such that for each $v \in V$, there exists a sequence $(v_k) \subset \mathcal{D}$ such that $v_k \rightarrow v$ and $\phi(v_k) \rightarrow \phi(v)$ as $k \rightarrow \infty$ [74, Lem. 3]. Since \mathcal{D} is countable, Theorem 23 ensures the existence of $\Omega'_1 \subset \Omega'$ with $\Omega'_1 \in \mathcal{F}'$ and $P'(\Omega'_1) = 1$ such that for each $v \in \mathcal{D}$ and all $\omega' \in \Omega'_1$, we have $\hat{\phi}_N(v, \omega') \rightarrow \phi(v)$. Fix $v \in V$ and let $(v_k) \subset \mathcal{D}$ be a sequence such that $v_k \rightarrow v$ and $\phi(v_k) \rightarrow \phi(v)$ as $k \rightarrow \infty$. We now proceed with a diagonalization argument (see, e.g., Corollary 1.16 or 1.18 in [3]). For each $k \in \mathbb{N}$ and every $\omega' \in \Omega'_1$, we have $\hat{\phi}_N(v_k, \omega') \rightarrow \phi(v_k)$ as $N \rightarrow \infty$. Moreover $\phi(v_k) \rightarrow \phi(v)$ as $k \rightarrow \infty$. Consequently, for each $\omega' \in \Omega'_1$, there exists a mapping $\mathbb{N} \ni N \mapsto k_{\omega'}(N) \in \mathbb{N}$ increasing to ∞ such that $\hat{\phi}_N(v_{k_{\omega'}(N)}, \omega') \rightarrow \phi(v)$ as $N \rightarrow \infty$. Since $v_k \rightarrow v$ as $k \rightarrow \infty$, we further have $v_{k_{\omega'}(N)} \rightarrow v$ as $N \rightarrow \infty$ for each $\omega' \in \Omega'_1$. Combining the derivations, we have shown that for each $\omega' \in \Omega'_1$ and every $v \in V$, there exists a sequence $(v_{k_{\omega'}(N)})$ converging to v as $N \rightarrow \infty$ and $\hat{\phi}_N(v_{k_{\omega'}(N)}, \omega') \rightarrow \phi(v)$ as $N \rightarrow \infty$. Since $\Omega'_0 \cap \Omega'_1 \in \mathcal{F}'$ and $P'(\Omega'_0 \cap \Omega'_1) = 1$, we have demonstrated the almost sure epiconvergence of $\hat{\phi}_N$ to ϕ . \square

B Law Invariance of $\text{AVaR}_\beta^\varepsilon$ and σ_β^ε

Both $\text{AVaR}_\beta^\varepsilon$ defined in (11) and σ_β^ε given in (12) are optimized certainty equivalents in the sense of [7], i.e. they are fully characterized by convex, continuous scalar regret functions $v_{\text{epi}, \varepsilon}, v_{\text{s}, \varepsilon} : \mathbb{R} \rightarrow \mathbb{R}$ such that for each $X \in L^2(\Omega, \mathcal{F}, P)$,

$$\begin{aligned} \text{AVaR}_\beta^\varepsilon[X] &= \inf_{t \in \mathbb{R}} \{ t + \mathbb{E}[v_{\text{epi}, \varepsilon}(X - t)] \}, \\ \sigma_\beta^\varepsilon[X] &= \inf_{t \in \mathbb{R}} \{ t + \mathbb{E}[v_{\text{s}, \varepsilon}(X - t)] \}, \end{aligned}$$

where $\varepsilon > 0$, $v_{\text{s}, \varepsilon}(x) := (1 - \beta)^{-1}(x)_\varepsilon^+$, and

$$v_{\text{epi}, \varepsilon}(x) := \begin{cases} -\frac{\varepsilon}{2} & \text{if } x \in (-\infty, -\varepsilon], \\ \frac{1}{2\varepsilon}x^2 + x & \text{if } x \in (-\varepsilon, \frac{\varepsilon\beta}{1-\beta}), \\ \frac{1}{1-\beta}(x - \frac{\varepsilon\beta^2}{2(1-\beta)}) & \text{otherwise.} \end{cases}$$

The fact that $\text{AVaR}_\beta^\varepsilon$ can be expressed in the above form has been demonstrated in Example 2 on p. 778 in [38].

It is not essential for the underlying probability space to be nonatomic for the law invariance of these functionals. Indeed, start by letting $v : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and hence, measurable. For each $X \in L^2(\Omega, \mathcal{F}, P)$ and $t \in \mathbb{R}$, let $v(X - t)$ be integrable, which is the case for both $v_{\text{epi}, \varepsilon}$ and $v_{s, \varepsilon}$. Let $X_1, X_2 \in L^2(\Omega, \mathcal{F}, P)$ be distributionally equivalent with respect to P . Since the distribution functions of X_1 and X_2 are equal and each distribution function uniquely determines a probability law on \mathbb{R} [10, Thm. 12.4], it holds that $P \circ X_1^{-1} = P \circ X_2^{-1}$. For all $t \in \mathbb{R}$, we have

$$\begin{aligned} \mathbb{E}[v(X_1 - t)] &= \int_{\Omega} v(X_1(\omega) - t) dP(\omega) = \int_{\mathbb{R}} v(x - t) dP \circ X_1^{-1}(x) \\ &= \int_{\mathbb{R}} v(x - t) dP \circ X_2^{-1}(x) = \int_{\Omega} v(X_2(\omega) - t) dP(\omega) = \mathbb{E}[v(X_2 - t)]. \end{aligned}$$

Hence, $\text{AVaR}_\beta^\varepsilon$ and σ_β^ε are law invariant. As a result, a large class of risk measures/optimized certainty equivalents are law invariant.

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