Introduction to the Theory of the Cosmic Microwave Background

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Acknowledgment

In the beginning of my work on this masters thesis we started with the idea of finding way to modify the CMBFast code to calculate the CMB anisotropy spectrum in an anisotropically expanding universe. As I studied the literature it turned out this was a highly non-trivial task and a full understanding of the CMB power spectrum in a ΛCDM universe was required. Hence it resulted in a full calculation as shown in this thesis.
I would like to thank my supervisor Professor Øvind Grøn for helpful discussion and insight. A special thanks goes to my family and friends who's support have been vital throughout the writing this thesis.

Contents

Chapter 1

Introduction

1.1 The New Cosmological Paradigm

The accidental[1] discovery of the *Cosmic Microwave Background* radiation by Penzias and Wilson[33] in 1965 started a new era in modern cosmology. Although it was already predicted by Gamow [15] as early as 1946, and one research group led by Dicke already [41] went looking for it, the discovery took the scientific community by surprise. In the Big Bang scenario, this radiation is thought to emanate from the young universe, at about 300.000 years after the Big Bang. But it was not until the launching of the COBE satellite in 1989 that precision measurements of this residual radiation could be made, showing a perfect thermal spectrum and a very small anisotropy in temperature of the sky of the order of one part in 100.000, which many would call noise. Since then ground based balloon experiments and the WMAP satellite have increased the precision of the measurements, and the upcoming PLANCK satellite will delve even further into the radiation, exploring the polarization spectrum amongst other aspects.
We start our exploration of the *Cosmic Microwave Background* (CMB) by studying the background cosmology given by a smoothly expanding universe. Throughout this thesis we will be using units of $c = \hbar = k_B = 1$.

1.2 The Smooth Background

We will here present the equations for the smooth background universe, which are often referred to as the Friedmann-Robertson-Walker (FRW) equations. We are throughout this work assuming a flat universe with coordinates $x^\mu =$

[1]Serendipitous as many authors have said lately.

(t, x, y, z) and with the metric

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{1.1}$$

where $a(t)$ is the expansion factor. We are assuming that the cosmological plasma is a perfect fluid, hence we will use $T_\mu^\nu = diag(-\rho, P, P, P)$, where ρ and P is the energy density and pressure respectively. The Einstein equations are given by $G_\mu^\nu = 8\pi G T_\mu^\nu$, where G_μ^ν is the Einstein tensor. Starting with equation 1.1 we obtain the FRW equations

$$H^2(t) = \frac{8\pi G}{3}\rho \tag{1.2}$$

$$\frac{1}{a}\frac{d^2a}{dt^2} + \frac{1}{2}H^2(t) = -4\pi G P. \tag{1.3}$$

We have here defined the *Hubble factor* $H = \frac{1}{a}\frac{da}{dt}$. It is at this time we introduce the *Hubble constant* $H_0 \equiv H(t_0)$ corresponding to the value of the Hubble factor today. The Hubble constant is parameterized as $H_0 = 100h\,\mathrm{km\,s^{-1}Mpc^{-1}}$, where current estimates for h is about 0.7. It is also convenient to set the the *critical density* $\rho_{cr} \equiv \frac{3H_0^2}{8\pi G}$, corresponding to the energy density of a flat universe. With these definitions we can write equation (1.2) as

$$H^2 = H_0^2\frac{\rho}{\rho_{cr}}. \tag{1.4}$$

Conservation of energy is upheld by the equation $T^\mu_{\nu;\mu} = 0$ which gives

$$\frac{d\rho}{dt} + 3H(\rho + P) = 0. \tag{1.5}$$

The expansion factor a is related to the cosmological redshift z by

$$1 + z(t) = \frac{a(t_0)}{a(t)}, \tag{1.6}$$

where t_0 is the present time.

1.3 Cosmological Species

In the standard model of cosmology we have basically 5 different constituents. These are the photons, baryons (including the electrons), neutrinos, (cold) dark matter and dark energy. We usually characterize these fluids by their equation of state $P = \omega\rho$, with $\omega = \frac{1}{3}$ for radiation and $\omega = 0$ for matter (dust). The simplest form of dark energy has $\omega = -1$, corresponding to a cosmological constant. Integration of equation (1.5) gives

$$\rho \propto a^{-3(1+\omega)}, \tag{1.7}$$

which leads to

$$\rho_{rad} \propto a^{-4} \quad , \text{Radiation} \tag{1.8}$$
$$\rho_m \propto a^{-3} \quad , \text{Matter} \tag{1.9}$$
$$\rho_{de} = \text{constant} \quad , \text{Cosmological Constant} \tag{1.10}$$

We employ present time normalization for the scale factor, i.e. we set $a(t_0) = 1$. With this convention we can write for species X

$$\rho_X = \rho_{0X} a^{-3(1+\omega)}, \tag{1.11}$$

where ρ_{0X} is the present time density. It has become customary to express the present day densities in terms of the critical density ρ_{cr}, i.e. we define $\Omega_X = \rho_{X0}/\rho_{cr}$. For the matter density we obtain

$$\rho_m = \rho_{cr}\Omega_{m0}a^{-3}, \tag{1.12}$$

which we could split up into a baryon part and a dark matter part

$$\rho_b = \rho_{cr}\Omega_b a^{-3} \tag{1.13}$$
$$\rho_{cdm} = \rho_{cr}\Omega_{cdm} a^{-3} \tag{1.14}$$

with $\Omega_{m0} = \Omega_b + \Omega_{cdm}$. For the photons we will use [10]

$$\rho_\gamma = 2\int \frac{d^3p}{(2\pi)^3}\frac{p}{e^{p/T}-1}, \tag{1.15}$$

where p is the photon momentum and T is the (zero order) Temperature. Calculating this integral [10] gives

$$\rho_\gamma = \frac{\pi^2}{15}T^4. \tag{1.16}$$

Observe that this shows that the temperature of the universe scales as a^{-1}, we can hence write

$$T = T_0 a^{-1}. \tag{1.17}$$

The present temperature of the CMB photons T_0 has been measured to great accuracy to be

$$T_0 = (2.725 \pm 0.002)K. \tag{1.18}$$

The CMB temperature of the sky is very uniform ($\sim 10^{-5}$) and the radiation has an almost perfect Planckan spectrum (see figure 1.1). We can use this value of T_0 to rewrite the the radiation energy density

$$\begin{aligned} \rho_r &= \rho_{cr}\, 2.47 \cdot 10^{-5} h^{-2} a^{-4} \\ &= \rho_{cr}\, \Omega_r a^{-4} \end{aligned} \tag{1.19}$$

In our flat universe we have that

$$\begin{aligned} \Omega_{Tot} &= \Omega_{m0} + \Omega_{de} + \Omega_r \\ &\simeq \Omega_{m0} + \Omega_{de} = 1, \end{aligned} \tag{1.20}$$

where we have approximated the total relative density Ω_{Tot} with only the dark energy and the matter components since radiation is completely negligible at t_0. Before we leave this section it is informative to state the scale factor in terms of cosmic time. This is

$$a \propto t^{\frac{2}{3(1+\omega)}}, \tag{1.21}$$

which one obtains from equation (1.2) using (1.3).

1.4 Cosmological Epochs

As we saw in the previous section, the different cosmological constituents decay at different rates, and radiation decays most rapidly. Hence one can expect that the universe undergoes epochs where the different species dominate. In the standard model the early universe was dominated by radiation which then goes over to a matter dominated universe. At later times the universe becomes less matter dominated, dark energy will overcome the matter dominance and the cosmos becomes dark energy dominated. An era which we will discover has importance in the generation of perturbations of the CMB is the time when the radiation density becomes equal to the matter density which is called the *epoch of matter-radiation equality.* The expansion factor $a_{eq} \equiv a(t_{eq})$ when this occurs is obtained by equating the matter and radiation densities. We obtain

$$a_{eq} = 4.15 \cdot 10^{-5} \left(\Omega_{m0} h^2\right)^{-1}. \tag{1.22}$$

It is interesting to calculate *when* this actually occurs in the ΛCDM model. One way to do this is to assume that the universe is completely radiation

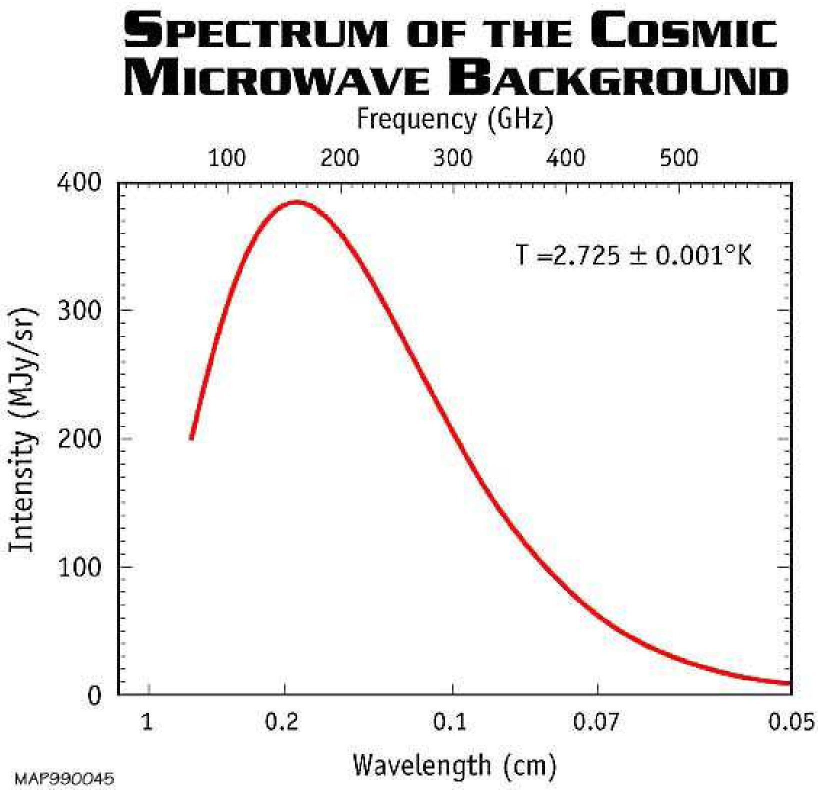

Figure 1.1: The CMB radiation spectrum given from the WMAP experiment. Where we to include the error bars, the thickness would be less than that of the curve. This figure is obtained from [36].

dominated up to the point of equality. We hence have $H = H_0\sqrt{\Omega_r}a^{-2}$. Since $dt = 1/(aH)da$, we get

$$\begin{aligned} t_{eq} = \int_0^{t_{eq}} dt &= \int_0^{a_{eq}} \frac{1}{aH_0\sqrt{\Omega_r}a^{-2}}da \\ &= \frac{1}{H_0\sqrt{\Omega_r}} \int_0^{a_{eq}} ada \\ &= \frac{a_{eq}^2}{2H_0\sqrt{\Omega_r}} \\ &= 4.28 \cdot 10^{-6} H_0^{-1} \simeq 59.000yr. \end{aligned} \tag{1.23}$$

Some authors do actually employ this value for the time of equality, but realistically this value must be incorrect as close to t_{eq} there is a significant amount of matter in the universe. To rectify this we would have to use $H = H_0\sqrt{\Omega_r a^{-4} + \Omega_{m0} a^{-3}}$ in the above integral, as in [34]. Firstly we have that $\rho_\gamma = \rho_{cr}\Omega_r a^{-4}$ and $\rho_m = \rho_{cr}\Omega_{mo} a^{-3}$. Then the scale factor at equality can be written as $a_{eq} = \Omega_r/\Omega_{mo}$. We thus obtain

$$\begin{aligned} t_{eq} &= \int_0^{a_{eq}} \frac{1}{aH_0\sqrt{\Omega_r a^{-4} + \Omega_{m0} a^{-3}}}\, da \\ &= \frac{1}{H_0\sqrt{\Omega_{m0}}} \int_0^{a_{eq}} \frac{a}{\sqrt{a_{eq} + a}}\, da \\ &= \frac{1}{H_0\sqrt{\Omega_{m0}}} \frac{2}{3} a_{eq}^{\frac{3}{2}} \left[2 - \sqrt{2}\right] \\ &= 3.35 \cdot 10^{-6} H_0^{-1} \\ &\simeq 47.000 yr. \end{aligned} \tag{1.24}$$

Hence the former value is about 20 % off. Of course, compared to the age of the universe this offset is insignificant, but it does carry a message that we must be careful when dealing with these matters. Specifically it is important after equality to include a radiation component even though matter is the dominant species.

1.4.1 The Cosmic Plasma

As the universe continues it's expansion, the baryonic part of matter is tightly coupled with the photons through Compton scattering creating what is known as the *photon-baryon plasma*. The perturbations occurring in the this fluid can be viewed [25] as an oscillator where the pressure from the photons tries to overcome the gravitational attraction. The presence of the baryons raises the inertia of the fluid increasing the effective mass of the oscillator. The (cold) dark matter components effect the photon perturbations only indirectly through their effect on the gravitational potential.

The oscillatory behavior of the cosmic fluid can only continue as long as the temperature is high enough. When the thermodynamical temperature decreases below the ionization energy for Hydrogen the electrons combine[2] with the protons and the photons are decoupled from the baryons, thus making the universe transparent. This era is known as the time of *recombination*, *decoupling* or *last scattering*. For the standard model this occurs for

$$a(t_{rec}) \equiv a_* \simeq 9.08 \cdot 10^{-4} \tag{1.25}$$

[2]Actually, detailed calculations show that this occurs roughly for $T \sim 0.25eV$, much less than the expected $13.6eV$.

which corresponds to a redshift of $z \simeq 1100$. During recombination the photons are freed from the baryons and are able to free stream in every direction. The photons encode the state of the cosmic fluid in the moments before last scattering, hence measuring this residual radiation probe the universe at the time of decoupling. This radiation is the CMB that we have mentioned often already. Figure 1.2 shows this radiation as observed from the WMAP experiment [36].

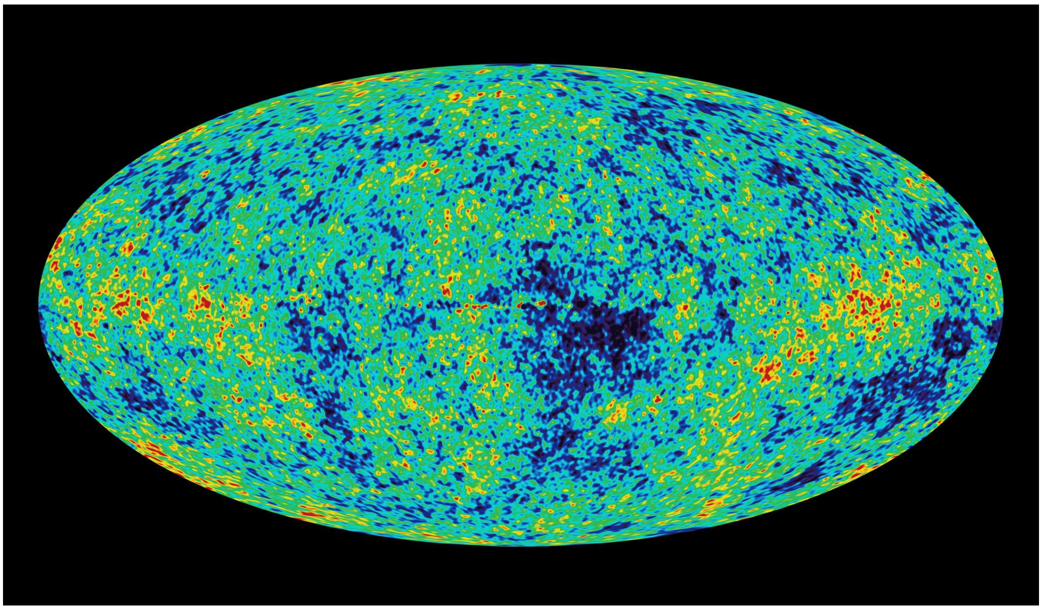

Figure 1.2: The Cosmic Microwave Background Radiation observed from the WMAP satellite [36]. The blue spots indicate hot regions while the red areas are cold.

1.5 Conformal Time

In an isotropically expanding universe it is both convenient and useful to introduce an alternative time scaling. This is known as *conformal time* η and is defined by

$$d\eta \equiv \frac{1}{a}dt = \frac{1}{a^2 H}da. \tag{1.26}$$

We will throughout this thesis mainly be working in the conformal time setting. The smooth metric of equation (1.1) becomes

$$ds^2 = a^2(\eta)\left(-d\eta^2 + dx^2 + dy^2 + dz^2\right). \tag{1.27}$$

We will study the physical interpretation of conformal time in chapter 3, for now it is sufficient to give an example of the conformal time in a universe with radiation and matter.

1.5.1 Conformal Time in a Radiation-Matter Universe

The conformal time in a matter-radiation universe can evaluated by integrating equation (1.26). With the Hubble factor given by

$$H = H_0\sqrt{\Omega_{mo}a^{-3} + \Omega_r a^{-4}}, \tag{1.28}$$

we obtain

$$\begin{aligned}\eta &= \frac{1}{H_0}\int_0^a dr \frac{1}{r^2\sqrt{\Omega_{mo}r^{-3} + \Omega_r r^{-4}}} \\ &= \frac{1}{H_0}\int_0^a dr \frac{1}{\sqrt{\Omega_{mo}r + \Omega_r}} \\ &= \frac{2\sqrt{\Omega_{m0}}}{H_0\Omega_{m0}}\left[\sqrt{a + \Omega_r/\Omega_{m0}} - \sqrt{\Omega_r/\Omega_{m0}}\right] \\ &= \frac{2}{H_0\sqrt{\Omega_{m0}}}\left[\sqrt{a + a_{eq}} - \sqrt{a_{eq}}\right].\end{aligned} \tag{1.29}$$

This is the conformal time in a cosmology with both matter and radiation. We will return to this expression in chapter 4.

1.6 The Goal

The aim of this thesis is to give a "not-so-short" introduction into the generation of the observed anisotropy of the Cosmic Microwave Background. We will throughout the chapters develop the tools and formalism necessary for this task. Whenever possible, full analytic derivations of results will be given. We are presenting the path of "least resistance" in understanding the CMB without sacrificing detail and restricting our treatment to a ΛCDM model. There will although some aspects we will not cover and we will make some simplifications in our treatment.

1.7 Overview

We start in chapter 2 by examining the Boltzmann equation of the cosmic plasma in the linear regime. We will here calculate the first few moments of the Boltzmann equation and state the Einstein equations for the perturbed metric. Here we find the fundamental equations for our problem.
In chapter 3 we look at Inflation. We will investigate a simple single-field inflationary model and see the implications this has on the generation of the primordial perturbations of the gravitational potential.
In chapter 4 we will make an in-depth study of the perturbations to the temperature and discover a useful approximation of the Boltzmann equations in the tight-coupling limit.
In chapter 5 we investigate the effects of inflation on structure formation and see how linear growth is treated in the literature.
Chapter 6 introduces the *power spectrum* of the CMB which quantifies the anisotropies in the background radiation. We will obtain an expression for the power spectrum in the large scale limit and an approximation in smaller scales.
Finally in chapter 7 we study our model for the CMB power spectrum and compare it to results obtained from CMB software. We also discuss some aspects for future work.
In the appendices we present some of the foundations of the formalism used throughout the thesis. In addition we include the numerical codes used in the simulation of some of the results.

Chapter 2

Cosmological Perturbation Theory

2.1 The Boltzmann Equation for the Photon

In the following we will be studying scalar mode perturbations to the metric. The Conformal Newton Gauge, also known as longitudinal gauge, first considered by Mukhanov [31] in 1992, is a gauge particularly suited for calculating the scalar perturbations. The line element in this gauge is given by

$$ds^2 = a^2(\eta)\left(-(1+2\Psi)d\eta^2 + (1+2\Phi)\,\delta_{ij}dx^i dx^j\right), \tag{2.1}$$

where we have introduced conformal time η with $dt = a\,d\eta$.
The potential Ψ can be interpreted as the gravitational potential in the Newtonian limit, and Φ is the fractional perturbation to the spatial curvature. We will see later that in the absence of anisotropic stress or pressure (see section 2.3), the potential $\Phi = -\Psi$. The aim of this chapter is to derive the Boltzmann equations for a relativistic cosmological fluid, specifically the photon. The strategy will be to find an expression for the geodesic equation in the given setting and combine these with the Boltzmann equation. The resulting equations will then be rewritten with spherical harmonic functions in Fourier space to obtain an infinite set of coupled differential equations.

2.1.1 The Distribution function in Phase space

At this point we introduce a six dimensional phase space consisting of the coordinates $(x^1, x^2, x^3, P_1, P_2, P_3)$, where x^i are the spatial coordinates and P_i are the conjugate momenta to x^i. From Hamiltons equations, the conjugate momenta are constant throughout the expansion[30]. We shall soon see how

these relate to the proper momenta p^i.
We now define the *distribution function* $f(x^i, P^i, t)$. This function gives the number of particles in the phase space volume element $dx^1dx^2dx^3dP^1dP^2dP^3$ as

$$dN = f(x^i, P^i, t)dx^1dx^2dx^3dP^1dP^2dP^3 \tag{2.2}$$

Note that f is a scalar[10] and invariant under canonical transformations.
For a relativistic bosonic gas, being in thermal equilibrium to 0'th order, the distribution function is of the form

$$f = \frac{g_s}{h^3}\frac{1}{exp\left(\frac{E-\mu}{k_bT}\right) - 1}, \tag{2.3}$$

where

- E is the particle energy.
- T is the temperature.
- μ is the chemical potential.
- h is the Planck's constant.
- k_b is the Boltzmann constant.
- g_s is the spin degrees of freedom.

We will set $h = k_b = 1$ from now on. For photons, $g_s = 2$, corresponding to the number of plane polarizations in orthogonal directions. The chemical potential μ can in fact be omitted in the expression for f. Just before nucleosynthesis the relativistic species consist of three neutrinos, photons, electrons and positrons. One can use the Standard Model to calculate the chemical potential for all species. The calculations imply that for successful nucleosynthesis one needs $\mu \ll T$. We will in the following only study the photon distribution function, thus we will set $E = p$, the photon momentum. Thus the modified form of the distribution function becomes

$$f = \frac{2}{exp\left(\frac{p}{T}\right) - 1} \tag{2.4}$$

Many interesting quantities can be expressed in terms of the distribution function. The most obvious one is the total particle number given by

$$N = \int f(x^i, P^i, \eta)\, dx^1dx^2dx^3dP^1dP^2dP^3. \tag{2.5}$$

More importantly from [29], we can express the energy-momentum tensor as

$$T_{\mu\nu} = \int dP_1 dP_2 dP_3 \, (-g)^{-1/2} \frac{P_\mu P_\nu}{P_0} f(x^i, P^i, \eta), \tag{2.6}$$

where g is the determinant of the matrix $(g_{\mu\nu})$. Observe now that in the following calculations I will omit the factor g_s , recalling to add it in again whenever necessary, f.eg when calculating $T_{\mu\nu}$.

2.1.2 The Geodesic Equations

Considering a space-time point $x^\mu = (\eta, \mathbf{x})$, the geodesic equation for a free particle is

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0, \tag{2.7}$$

where λ is an affine parameter. I will now rewrite the equation in terms of the 4-momentum P^μ.
By definition we have that

$$P^\mu \equiv \frac{dx^\mu}{d\lambda}, \tag{2.8}$$

so the natural choice for λ would be

$$P^0 = \frac{dx^0}{d\lambda} = \frac{d\eta}{d\lambda}. \tag{2.9}$$

We insert these definitions into the geodesic equation to get an expression involving only P^μ

$$P^0 \frac{dP^\mu}{d\eta} + \Gamma^\mu_{\alpha\beta} P^\alpha P^\beta = 0. \tag{2.10}$$

Using the expression for the Christoffel symbols, this equation can be treated further [10] to obtain

$$\frac{dP^\mu}{d\eta} = g^{\mu\nu} \left(\frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x^\nu} - \frac{\partial g_{\nu\alpha}}{\partial x^\beta} \right) \frac{P^\alpha P^\beta}{P^0}, \tag{2.11}$$

where $g_{\mu\nu}$ is the metric tensor.
We shall here derive the geodesic equation for the photon which we will use in conjunction with the Boltzmann equation in the next section. But we gain some insight into dynamics of our system by studying the geodesic equation alone. This will take us into the Sachs-Wolfe effect [35] in a simplified form.

The 4-momentum P^μ

We need to find an expression for the 4-momentum P^μ of the photon. Note first that since the photon is massless, we have

$$P^2 \equiv g_{\mu\nu} P^\mu P^\nu = 0, \tag{2.12}$$

that is

$$g_{00}(P^0)^2 + g_{ij} P^i P^j = 0. \tag{2.13}$$

Defining $p^2 = g_{ij} P^i P^j$ and inserting the value for g_{00} yields

$$-a^2(1+2\Psi)(P^0)^2 + p^2 = 0, \tag{2.14}$$

which when solving for P^0 leads to

$$P^0 = \frac{p}{a\sqrt{1+2\Psi}} \approx (1-\Psi)\frac{p}{a}\,. \tag{2.15}$$

To find the comoving momentum P^i we note that it is proportional to the proper momentum given by $p^i = p\,n^i$ where $n^i = n_i$ is a unit directional vector

$$P^i = D \cdot p^i. \tag{2.16}$$

Hence we have that

$$\begin{aligned} p^2 &= g_{ij} P^i P^j \\ &= g_{ij}\, p^2 n^i n^j D^2 \\ &= a^2(1+2\Phi)\, p^2 \delta_{ij} n^i n^j D^2 \\ &= a^2(1+2\Phi)\, p^2 D^2, \end{aligned} \tag{2.17}$$

where I have utilized the relation $\delta_{ij} n^i n^j = 1$, which holds since n^i is a unit vector. Solving for D in equation (2.17) we obtain

$$D = \frac{1}{a\sqrt{1+2\Phi}} \approx \frac{1-\Phi}{a} \tag{2.18}$$

Hence the comoving momentum P^i is given by

$$P^i = p\, n^i \frac{1-\Phi}{a} \tag{2.19}$$

We will use these relations in the following section to find the geodesic equation for the photon. Before we proceed to do this, let us find an expression for the velocity of the photon $\frac{dx^i}{d\eta}$. By the chain rule, we have that

$$\begin{aligned} \frac{dx^i}{d\eta} &= \frac{dx^i}{d\lambda}\frac{d\lambda}{d\eta} \\ &= \frac{P^i}{P^0}. \end{aligned} \tag{2.20}$$

Inserting equations (2.15) and (2.19) into this expression we obtain to first order in Ψ and Φ

$$\frac{dx^i}{d\eta} = n^i(1 - \Phi + \Psi). \tag{2.21}$$

The Geodesic Equation for the Photon

We will now find an expression for the Geodesic equation in terms of the variables introduced here. It will suffice to calculate the time component since this will give us directly the change in p , i.e. $\frac{dp}{d\eta}$. In the following, over-dots will represent differentiation with respect to conformal time.For easy reference we present some useful expressions we will use in the following calculations

$$g_{00} = -a^2(1 + 2\Psi) \tag{2.22}$$

$$g^{00} = \frac{-1 + 2\Psi}{a^2} \tag{2.23}$$

$$g_{ij} = a^2\delta_{ij}(1 + 2\Phi) \tag{2.24}$$

$$\frac{\partial g_{00}}{\partial \eta} = -2a\dot{a}(1 + 2\Psi) - 2a^2\frac{\partial \Psi}{\partial \eta} \tag{2.25}$$

$$\frac{\partial g_{00}}{\partial x^i} = -2a^2\frac{\partial \Psi}{\partial x^i} \tag{2.26}$$

$$\frac{\partial g_{ij}}{\partial \eta} = \delta_{ij}\left(2a\dot{a}(1 + 2\Phi) + 2a^2\frac{\partial \Phi}{\partial \eta}\right) \tag{2.27}$$

$$\delta_{ij}\frac{P^iP^j}{P^0} = \frac{p}{a}(1 - 2\Phi + \Psi) \tag{2.28}$$

$$\mathcal{H} \equiv \frac{\dot{a}}{a}. \tag{2.29}$$

A few comments are in order at this point. Equation (2.29) defines a 'conformal Hubble parameter' related to the *Hubble* parameter H by $\mathcal{H} = aH$. Equation (2.23) follows from $g^{00} = \frac{1}{g_{00}}$ since $g_{\mu\nu}$ is diagonal.

Concerning equation (2.28), we use equations (2.15) and (2.19) to obtain

$$\begin{aligned}
\delta_{ij}\frac{P^iP^j}{P^0} &= \frac{p^2(1-2\Phi)/a^2}{p(1-\Psi)/a} \\
&\approx \frac{p}{a}(1-2\Phi)(1+\Psi) \\
&\approx \frac{p}{a}(1-2\Phi+\Psi).
\end{aligned} \tag{2.30}$$

We can now undertake the task set before us. The time component of the geodesic equation is

$$\begin{aligned}
\frac{dP^0}{d\eta} &= g^{0\nu}\left(\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\nu} - \frac{\partial g_{\nu\alpha}}{\partial x^\beta}\right)\frac{P^\alpha P^\beta}{P^0} \\
&= g^{0\nu}\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\frac{P^\alpha P^\beta}{P^0} - g^{0\nu}\frac{\partial g_{\nu\alpha}}{\partial x^\beta}\frac{P^\alpha P^\beta}{P^0}\,.
\end{aligned} \tag{2.31}$$

Starting with the right hand side, the first term is

$$\begin{aligned}
\frac{1}{2}g^{0\nu}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\frac{P^\alpha P^\beta}{P^0} &= \frac{1}{2}g^{00}\frac{\partial g_{\alpha\beta}}{\partial x^0}\frac{P^\alpha P^\beta}{P^0} \\
&= \frac{1}{2}g^{00}\left(\frac{\partial g_{00}}{\partial\eta}\frac{P^0P^0}{P^0} + \frac{\partial g_{ij}}{\partial\eta}\frac{P^iP^j}{P^0}\right) \\
&= \frac{1}{2}g^{00}g_{00,0}P^0 + \frac{1}{2}g^{00}g_{ij,0}\frac{P^iP^j}{P^0} \\
&= \frac{-1+2\Psi}{2a^2}\left(-2a\dot{a}(1+2\Psi) - 2a^2\frac{\partial\Psi}{\partial\eta}\right)P^0 \\
&\quad + \frac{-1+2\Psi}{2a^2}\left(2a\dot{a}(1+2\Phi) + 2a^2\frac{\partial\Phi}{\partial\eta}\right)\delta_{ij}\frac{P^iP^j}{P^0} \\
&= \frac{\dot{a}}{a}(1-2\Psi)(1+2\Psi)P^0 + \frac{\partial\Psi}{\partial\eta}P^0 \\
&\quad - \left(\frac{\dot{a}}{a}(1-2\Psi)(1+2\Phi) + \frac{\partial\Phi}{\partial\eta}\right)\delta_{ij}\frac{P^iP^j}{P^0}\,.
\end{aligned} \tag{2.32}$$

Recall that we omit all quadratic terms in Φ and Ψ. Hence

$$\frac{\partial\Psi}{\partial\eta}(1-2\Psi)P^0 \to \frac{\partial\Psi}{\partial\eta}P^0, \tag{2.33}$$

and $(1-2\Psi)(1+2\Psi) = 1-4\Psi^2 \approx 1$ and $(1-2\Psi)(1+2\Phi) \approx 1-2\Psi+2\Psi$. Inserting this and equations (2.28), (2.29) and (2.15) into equation (2.32) we get

$$
\begin{aligned}
\frac{1}{2}g^{0\nu}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\frac{P^\alpha P^\beta}{P^0} &= \frac{\dot a}{a}P^0 + \frac{\partial\Psi}{\partial\eta}P^0 \\
&\quad - \left(\frac{\dot a}{a}(1-2\Psi+2\Phi) + \frac{\partial\Phi}{\partial\eta}\right)\delta_{ij}\frac{P^iP^j}{P^0} \\
&= \mathcal{H}\frac{p}{a}(1-\Psi) + \frac{p}{a}\frac{\partial\Psi}{\partial\eta} \\
&\quad - \left(\mathcal{H}(1-2\Psi+2\Phi) + \frac{\partial\Phi}{\partial\eta}\right)\frac{p}{a}(1-2\Phi+\Psi) \\
&= \mathcal{H}\frac{p}{a}(1-\Psi) + \frac{p}{a}\frac{\partial\Psi}{\partial\eta} - \mathcal{H}\frac{p}{a}(1-\Psi) - \frac{p}{a}\frac{\partial\Phi}{\partial\eta} \\
&= \frac{p}{a}\frac{\partial\Psi}{\partial\eta} - \frac{p}{a}\frac{\partial\Phi}{\partial\eta}\,.
\end{aligned}
\tag{2.34}
$$

In the third line I have used that

$$
(1+\Psi-2\Phi)\cdot(1-2\Psi+2\Phi) = 1-2\Psi+2\Phi+\Psi-2\Phi = 1-\Psi. \tag{2.35}
$$

Working now with the second term gives

$$
\begin{aligned}
g^{0\nu}\frac{\partial g_{\nu\alpha}}{\partial x^\beta}\frac{P^\alpha P^\beta}{P^0} &= g^{00}\frac{\partial g_{00}}{\partial x^\beta}P^\beta \\
&= g^{00}g_{00,0}P^0 + g^{00}g_{00,i}P^i \\
&= -\frac{1-2\Psi}{a^2}\left(-2a\dot a(1+2\Psi) - 2a^2\frac{\partial\Psi}{\partial\eta}\right)P^0 \\
&\quad - \frac{1-2\Psi}{a^2}\left(-2a^2\frac{\partial\Psi}{\partial x^i}\right)P^i \\
&= 2\mathcal{H}P^0 + 2\frac{\partial\Psi}{\partial\eta}P^0 + 2\frac{\partial\Psi}{\partial x^i}P^i \\
&= 2\mathcal{H}\frac{p}{a}(1-\Psi) + 2\frac{p}{a}\frac{\partial\Psi}{\partial\eta} + 2\frac{\partial\Psi}{\partial x^i}\frac{p}{a}n^i\,.
\end{aligned}
\tag{2.36}
$$

Subtracting equation (2.36) from (2.34), the right hand side of the Geodesic equation (2.31) becomes

$$
-2\mathcal{H}\frac{p}{a}(1-\Psi) - \frac{p}{a}\frac{\partial\Psi}{\partial\eta} - \frac{p}{a}\frac{\partial\Phi}{\partial\eta} - 2\frac{\partial\Psi}{\partial x^i}\frac{p}{a}n^i\,. \tag{2.37}
$$

For the left hand side of equation (2.31) we have that

$$\begin{aligned}
\frac{dP^0}{d\eta} &= \frac{d}{d\eta}\left(p(1-\Psi)\frac{1}{a}\right) \\
&= \frac{1}{a}\frac{dp}{d\eta}(1-\Psi) - \frac{p}{a}\frac{d\Psi}{d\eta} + p(1-\Psi)\frac{d}{d\eta}\left(\frac{1}{a}\right) \\
&= \frac{1}{a}\frac{dp}{d\eta}(1-\Psi) - \frac{p}{a}\left(\frac{\partial\Psi}{\partial\eta} + \frac{\partial\Psi}{\partial x^i}\frac{dx^i}{d\eta}\right) - \frac{p}{a}(1-\Psi)\mathcal{H} \\
&= \frac{1}{a}\frac{dp}{d\eta}(1-\Psi) - \frac{p}{a}\frac{\partial\Psi}{\partial\eta} - \frac{p}{a}\frac{\partial\Psi}{\partial x^i}n^i - \frac{p}{a}(1-\Psi)\mathcal{H}\,, \qquad (2.38)
\end{aligned}$$

where in the last line I have used $\frac{dx^i}{d\eta} = n^i(1-\Phi+\Psi)$. Combining equations (2.38) and (2.37) we obtain

$$\begin{aligned}
\frac{1}{a}\frac{dp}{d\eta}(1-\Psi) - \frac{p}{a}\frac{\partial\Psi}{\partial\eta} - \frac{p}{a}\frac{\partial\Psi}{\partial x^i}n^i - \frac{p}{a}(1-\Psi)\mathcal{H} &= -2\mathcal{H}\frac{p}{a}(1-\Psi) - \frac{p}{a}\frac{\partial\Psi}{\partial\eta} \\
&\quad - \frac{p}{a}\frac{\partial\Phi}{\partial\eta} - 2\frac{\partial\Psi}{\partial x^i}\frac{p}{a}n^i, \quad (2.39)
\end{aligned}$$

which after rearrangement and multiplying by $(1+\Psi)$ yields

$$\boxed{\frac{1}{p}\frac{dp}{d\eta} = -\mathcal{H} - \frac{\partial\Phi}{\partial\eta} - n^i\frac{\partial\Psi}{\partial x^i}} \qquad (2.40)$$

This is the geodesic equation for the photon, which is quite interesting to study by itself. It describes the change in the photon energy in an expanding universe boils down to three physical mechanisms. The first term quantifies the loss of energy due to the expansion. The second term can be interpreted as a shift in the energy because of a time varying gravitational potential, while the last term is ascribed to a spatial variation of gravity due to matter density variations, as photons lose energy climbing out of potential wells. This is basically the first encounter with the *Sachs-Wolfe effect* [35], which we will see many manifestations of throughout this work.
We will now continue with our work by combining our newly acquired equation with the Boltzmann equation.

2.1.3 The Boltzmann Equation

The evolution of the distribution function is given by the Boltzmann equation (see appendix A)

$$\frac{df}{dt} = C[f], \qquad (2.41)$$

where $C[f]$ is the collision term with other species, mainly with the electrons through Thompson scattering. In the absence of collisional effects, the above equation is often referred to as the *Liouville equation.* Introducing conformal time η we get

$$\frac{df}{dt} = \frac{d\eta}{dt}\frac{df}{d\eta} = \frac{1}{a}\frac{df}{d\eta}. \tag{2.42}$$

Thus equation (2.41) can be written as

$$\frac{df}{d\eta} = a \cdot C[f]. \tag{2.43}$$

We will now study only the left hand side of (2.43), including the collision term in the next section. Recall that the distribution function is now a function of x^i, p and n^i. Hence the total derivative $\frac{df}{d\eta}$ is expanded as

$$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + \frac{dx^i}{d\eta}\frac{\partial f}{\partial x^i} + \frac{dp}{d\eta}\frac{\partial f}{\partial p} + \frac{dn^i}{d\eta}\frac{\partial f}{\partial n^i}. \tag{2.44}$$

Inserting equation (2.21) on page 23 and the geodesic equation (2.40) we get

$$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + n^i\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left[\mathcal{H} + \frac{\partial \Phi}{\partial \eta} + n^i\frac{\partial \Psi}{\partial x^i}\right] + \frac{dn^i}{d\eta}\frac{\partial f}{\partial n^i} \tag{2.45}$$

We can at this point simplify equation (2.45) by omitting the last term. In the unperturbed universe, the distribution function f is a function of only p and not the photon direction n^i. Thus $\frac{\partial f}{\partial n^i}$ is non-zero only for higher order perturbations, i.e. 1.order or higher. In addition, the change in photon direction, $\frac{dn^i}{d\eta}$, is non-zero only in the presence of the potentials Ψ and Φ. Otherwise the photon moves in a straight line. Hence the product $\frac{dn^i}{d\eta}\frac{\partial f}{\partial n^i}$ is at least of a 2.order term in the potentials thus rendering it negligible in our linear setting[25]. This leaves us the expression

$$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + n^i\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left[\mathcal{H} + \frac{\partial \Phi}{\partial \eta} + n^i\frac{\partial \Psi}{\partial x^i}\right]. \tag{2.46}$$

2.1.4 The Brightness Function

The *Brightness function* of the photon [28] is defined as

$$\Theta = \Theta(\mathbf{x}, \mathbf{n}, \eta) \equiv \frac{\delta T}{T}, \tag{2.47}$$

i.e, the fractional temperature fluctuation. Observe that we have defined Θ explicitly as a function of $\mathbf{x}$, $\mathbf{n}$, η and not of the magnitude of the momentum

p. This will turn out to be a valid assumption, following directly from that p is virtually unchanged through interaction with other species', specifically the photons through Compton scattering. The distribution function can now be written as

$$f(\mathbf{x}, p, \mathbf{n}, \eta) = \frac{1}{exp\left[\frac{p}{T(\eta)(1+\Theta)}\right] - 1}. \tag{2.48}$$

Observe that we have set the (zero order) temperature T as a function of time and not space. In the zero order universe, the photons are homogeneously distributed, rendering T to be independent of position $\mathbf{x}$ and direction $\mathbf{n}$. Since the perturbation Θ is small (of the order 10^{-5}), we can Taylor expand the distribution function f to first order in Θ. Defining

$$f_0 = \frac{1}{exp\left(\frac{p}{T}\right) - 1}, \tag{2.49}$$

as the zero order distribution function. Expanding equation (2.48) we obtain

$$\begin{aligned} f &\approx f_0 + \frac{\partial f_0}{\partial T} \cdot T\Theta \\ &= \frac{1}{exp\left(\frac{p}{T}\right) - 1} + \frac{\partial}{\partial T}\left(f_0\right) \cdot T\Theta. \end{aligned} \tag{2.50}$$

We will be using equation (2.50) with Boltzmann Equation (2.46). It will thus be handy to express the derivative $\frac{\partial}{\partial T}$ by a derivative with respect to the momentum p since the derivative $\frac{\partial}{\partial p}$ is present in the Boltzmann Equation. We can accomplish this by noting that

$$\begin{aligned} T\frac{\partial f_0}{\partial T} &= T \cdot \frac{1}{\left(exp\left(\frac{p}{T}\right) - 1\right)^2} \cdot exp\left(\frac{p}{T}\right) \cdot \frac{p}{T^2} \\ &= \frac{exp\left(\frac{p}{T}\right)}{\left(exp\left(\frac{p}{T}\right) - 1\right)^2} \cdot \frac{p}{T} \\ &= -p\frac{\partial f_0}{\partial p}. \end{aligned} \tag{2.51}$$

We can use equation (2.51) to interchange derivatives whenever appropriate. Hence we obtain

$$f = f_0 - p\frac{\partial f_0}{\partial p}\Theta. \tag{2.52}$$

We will soon insert this equation into the Boltzmann equation in section 2.1.6, but let us first investigate the zeroth order Boltzmann equation.

2.1.5 Zero Order Temperature Scaling

We will now see that we can extract interesting information about our universe by just studying the Boltzmann equation up to zeroth order, which will also serve as a consistency check for our work. To zero order, all instances of Φ, Ψ and Θ will vanish. Thus the equation (2.46), the zeroth order Boltzmann equation becomes

$$\frac{df}{d\eta} = \frac{\partial f}{\partial \eta} + n^i \frac{\partial f}{\partial x^i} - p \frac{\partial f}{\partial p} \mathcal{H} = a\, C[f]_0 \,, \tag{2.53}$$

where I have now included the (zeroth order) collision term $C[f]_0$. Inserting the expression for the distribution, $f = f_0$, and recalling that f_0 is independent of x_i, we get

$$\frac{\partial f_0}{\partial \eta} - p \frac{\partial f}{\partial p} \mathcal{H} = a\, C[f]_0 \,. \tag{2.54}$$

The zeroth order collision term is 0, since we have set the distribution function to a Bose-Einstein distribution where the photons are in equilibrium. Thus the collision effects will by definition cancel [10]. Hence we are left with

$$\frac{\partial f_0}{\partial \eta} - p \frac{\partial f}{\partial p} \mathcal{H} = 0 \,. \tag{2.55}$$

Using equation (2.51), we have that

$$\begin{aligned} \frac{\partial f_0}{\partial \eta} &= \frac{\partial f_0}{\partial T} \frac{dT}{d\eta} \\ &= -\frac{p}{T} \frac{\partial f_0}{\partial p} \frac{dT}{d\eta} \end{aligned} \tag{2.56}$$

which gives

$$\left[-\frac{1}{T} \frac{dT}{d\eta} - \mathcal{H} \right] \frac{\partial f_0}{\partial p} = 0 \,. \tag{2.57}$$

This is equivalent to

$$-\frac{1}{T} \frac{dT}{d\eta} = \frac{1}{a} \frac{da}{d\eta} \,. \tag{2.58}$$

or simply

$$-\frac{dT}{T} = \frac{da}{a} \,. \tag{2.59}$$

This implies that

$$T \sim \frac{1}{a} \,, \tag{2.60}$$

which is exactly what we expect, a result previously obtained from heuristic arguments in chapter 1. Hence the temperature of the universe T scales as a^{-1}.

2.1.6 First Order Boltzmann Equation

We will now continue our derivation of the Boltzmann equation of the photon, going now to first order in Φ, Ψ and Θ. Inserting equation (2.52) for the distribution function into equation (2.46) we obtain (recall that Θ is independent of p)

$$\begin{aligned}\frac{df}{d\eta} &= \frac{\partial}{\partial\eta}\left(f_0 - p\frac{\partial f_0}{\partial p}\Theta\right) + n^i\frac{\partial}{\partial x^i}\left(f_0 - p\frac{\partial f_0}{\partial p}\Theta\right)\\ &\quad - p\frac{\partial}{\partial p}\left(f_0 - p\frac{\partial f_0}{\partial p}\Theta\right)\times\left[\mathcal{H} + \frac{\partial\Phi}{\partial\eta} + n^i\frac{\partial\Psi}{\partial x^i}\right]\\ &= \frac{\partial f_0}{\partial\eta} - p\frac{\partial}{\partial\eta}\left[\frac{\partial f_0}{\partial p}\Theta\right] + n^i\frac{\partial f_0}{\partial x^i} - p\,n^i\frac{\partial f_0}{\partial p}\frac{\partial\Theta}{\partial x^i}\\ &\quad - p\frac{\partial f_0}{\partial p}\mathcal{H} + p\,\Theta\frac{\partial}{\partial p}\left[p\frac{\partial f_0}{\partial p}\right]\mathcal{H} - p\frac{\partial f_0}{\partial p}\left[\frac{\partial\Phi}{\partial\eta} + n^i\frac{\partial\Psi}{\partial x^i}\right].\end{aligned} \tag{2.61}$$

The first and the fifth term cancel as shown in section 2.1.5 . The third term is 0 since f_0 has no explicit position dependence. Thus

$$\begin{aligned}\frac{df}{d\eta} &= -p\frac{\partial}{\partial\eta}\left[\frac{\partial f_0}{\partial p}\Theta\right] - p\,n^i\frac{\partial f_0}{\partial p}\frac{\partial\Theta}{\partial x^i}\\ &\quad + p\,\Theta\mathcal{H}\frac{\partial}{\partial p}\left[p\frac{\partial f_0}{\partial p}\right] - p\frac{\partial f_0}{\partial p}\left[\frac{\partial\Phi}{\partial\eta} + n^i\frac{\partial\Psi}{\partial x^i}\right].\end{aligned} \tag{2.62}$$

Now we have that

$$-p\frac{\partial}{\partial\eta}\left[\frac{\partial f_0}{\partial p}\Theta\right] = -p\,\Theta\frac{\partial}{\partial\eta}\frac{\partial f_0}{\partial p} - p\frac{\partial f_0}{\partial p}\frac{\partial\Theta}{\partial\eta}. \tag{2.63}$$

Inserting a temperature derivative and using $\frac{\partial f_0}{\partial T} = -\frac{p}{T}\frac{\partial f_0}{\partial p}$, we get

$$\begin{aligned}-p\frac{\partial}{\partial\eta}\left[\frac{\partial f_0}{\partial p}\Theta\right] &= -p\,\Theta\frac{dT}{d\eta}\frac{\partial}{\partial T}\frac{\partial f_0}{\partial p} - p\frac{\partial f_0}{\partial p}\frac{\partial\Theta}{\partial\eta}\\ &= p\,\Theta\frac{dT/d\eta}{T}\frac{\partial}{\partial p}\left(p\frac{\partial f_0}{\partial p}\right) - p\frac{\partial f_0}{\partial p}\frac{\partial\Theta}{\partial\eta}\\ &= -p\,\Theta\frac{da/d\eta}{a}\frac{\partial}{\partial p}\left(p\frac{\partial f_0}{\partial p}\right) - p\frac{\partial f_0}{\partial p}\frac{\partial\Theta}{\partial\eta}\\ &= -p\,\Theta\mathcal{H}\frac{\partial}{\partial p}\left(p\frac{\partial f_0}{\partial p}\right) - p\frac{\partial f_0}{\partial p}\frac{\partial\Theta}{\partial\eta},\end{aligned} \tag{2.64}$$

where I have used that $\frac{dT/d\eta}{T} = -\frac{da/d\eta}{a}$. Inserting this result into equation (2.62) we get

$$\frac{df}{d\eta} = -p\,\Theta\mathcal{H}\frac{\partial}{\partial p}\left(p\frac{\partial f_0}{\partial p}\right) - p\frac{\partial f_0}{\partial p}\frac{\partial \Theta}{\partial \eta} - p\,n^i\frac{\partial f_0}{\partial p}\frac{\partial \Theta}{\partial x^i}$$
$$+ p\,\Theta\mathcal{H}\frac{\partial}{\partial p}\left(p\frac{\partial f_0}{\partial p}\right) - p\frac{\partial f_0}{\partial p}\left[\frac{\partial \Phi}{\partial \eta} + n^i\frac{\partial \Psi}{\partial x^i}\right]. \tag{2.65}$$

which leads to

$$\frac{df}{d\eta} = -p\frac{\partial f_0}{\partial p}\left[\frac{\partial \Theta}{\partial \eta} + n^i\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial \eta} + n^i\frac{\partial \Psi}{\partial x^i}\right]. \tag{2.66}$$

Thus we finally obtain

$$-p\frac{\partial f_0}{\partial p}\left[\frac{\partial \Theta}{\partial \eta} + n^i\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial \eta} + n^i\frac{\partial \Psi}{\partial x^i}\right] = a\,C[f]\,. \tag{2.67}$$

This is the first order Boltzmann Equation, which we will spend a great deal of time studying in the coming sections.

2.1.7 Fourier Convention

Much of the analysis in this work will be done in Fourier space. Hence we need to state the Fourier convention to be used here. Firstly, the three dimensional Fourier integral of a function $H(\vec{x})$ is defined to be

$$\tilde{H}(\vec{k}) = \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}H(\vec{x}) \tag{2.68}$$

with inverse

$$H(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\tilde{H}(\vec{k}) \tag{2.69}$$

The Fourier modes will be characterized by the wave vector magnitude $k = \sqrt{\vec{k}\cdot\vec{k}}$. Hence $\frac{\partial}{\partial x^j} \to i\,k^j$. Note that k^i is an Euclidian 3D vector, so $k^i = k_i$.

2.1.8 Free Streaming with no collisions

As a short illustration of the meaning of equation (2.67), I will now study the Boltzmann equation (2.66) when the collision term $C[f]$ is set to zero. This corresponds to a time right after recombination where the photons free stream towards us. We get

$$\frac{\partial \Theta}{\partial \eta} + n^i\frac{\partial \Theta}{\partial x^i} = -\frac{\partial \Phi}{\partial \eta} - n^i\frac{\partial \Psi}{\partial x^i}. \tag{2.70}$$

We now go to Fourier space (see section 2.1.7). We obtain

$$\dot{\tilde{\Theta}} + i\,k\mu\,\tilde{\Theta} = -\dot{\tilde{\Phi}} - i\,k\mu\,\tilde{\Psi}, \tag{2.71}$$

where we have introduced

$$\mu \equiv \frac{\vec{k}\cdot\vec{n}}{k} \tag{2.72}$$

which is nothing else than the cosine of the angle between the wave vector and the photon direction[1] n. The Boltzmann equation can be written as

$$e^{-ik\mu\eta}\frac{\partial}{\partial\eta}\left(\tilde{\Theta}e^{ik\mu\eta}\right) = -\dot{\tilde{\Phi}} - i\,k\mu\,\tilde{\Psi}, \tag{2.73}$$

which implies

$$\begin{aligned}
\frac{\partial}{\partial\eta}\left(\tilde{\Theta}e^{ik\mu\eta}\right) &= -\dot{\tilde{\Phi}}e^{ik\mu\eta} - i\,k\mu\,\tilde{\Psi}e^{ik\mu\eta} \\
&= -\dot{\tilde{\Phi}}e^{ik\mu\eta} - \tilde{\Psi}\frac{\partial}{\partial\eta}\left(e^{ik\mu\eta}\right) \\
&= -\dot{\tilde{\Phi}}e^{ik\mu\eta} + \dot{\tilde{\Psi}}e^{ik\mu\eta} - \frac{\partial}{\partial\eta}\left(\tilde{\Psi}e^{ik\mu\eta}\right) \\
&= \frac{\partial}{\partial\eta}\left(\tilde{\Psi}-\tilde{\Phi}\right)e^{ik\mu\eta} - \frac{\partial}{\partial\eta}\left(\tilde{\Psi}e^{ik\mu\eta}\right).
\end{aligned} \tag{2.74}$$

In the absence of anisotropic stress, we have that $\tilde{\Psi} = -\tilde{\Phi}$ (see section 2.3). Hence after rearrangement we have

$$\frac{\partial}{\partial\eta}\left(\tilde{\Theta}e^{ik\mu\eta}\right) + \frac{\partial}{\partial\eta}\left(\tilde{\Psi}e^{ik\mu\eta}\right) = 2\frac{\partial\tilde{\Psi}}{\partial\eta}e^{ik\mu\eta}. \tag{2.75}$$

We can easily integrate this equation

$$\int_{\eta_*}^{\eta_0}\frac{\partial}{\partial\eta}\left(\left(\tilde{\Theta}+\tilde{\Psi}\right)e^{ik\mu\eta}\right)d\eta = \int_{\eta_*}^{\eta_0}2\frac{\partial\tilde{\Psi}}{\partial\eta}e^{ik\mu\eta}d\eta, \tag{2.76}$$

where η_* is the time of recombination and η_0 is the present time. Integration yields

$$\left[\left(\tilde{\Theta}+\tilde{\Psi}\right)e^{ik\mu\eta}\right]_{\eta_*}^{\eta_0} = \int_{\eta_*}^{\eta_0}2\frac{\partial\tilde{\Psi}}{\partial\eta}e^{ik\mu\eta}d\eta, \tag{2.77}$$

[1]Note that for $\mu = 1$, the photon is aligned with $\vec{k}$ thus traveling along a direction of changing temperature. If $\mu = 0$, the photon is moving along a direction of non-changing temperature[10].

which after rearrangement gives

$$\tilde{\Theta}(\eta_0) + \tilde{\Psi}(\eta_0) = \left(\tilde{\Theta}(\eta_*) + \tilde{\Psi}(\eta_*)\right) e^{ik\mu(\eta_* - \eta_0)} + e^{-ik\mu\eta_0} \int_{\eta_*}^{\eta_0} 2\frac{\partial \tilde{\Psi}}{\partial \eta} e^{ik\mu\eta} d\eta. \tag{2.78}$$

Equation (2.78) represents the anisotropies in the CMB temperature field observed today in terms of the anisotropies at recombination. The exponential factors represents phase shifts in the wave fronts which I will neglect at this stage.
Overdense regions are signified by $\tilde{\Psi} < 0$, hence the observed temperature fluctuations (the lhs of eq.(2.78)) at these regions will seem colder. This corresponds to photons being redshifted as they climb out of potential wells. Conversely, under dense regions will seem hotter. This is the *Sachs-Wolfe effect* in a simplified form. In addition we see from the above equation that there will be an additional change in the spectrum if the potential $\tilde{\Psi}$ is changing through the free streaming, quantified by the integral. This is due to a change in the photon energy as it passes through time-varying potentials. This is the *Integrated Sachs-Wolfe effect*, hence the name.

2.1.9 The Collision Term $C[f]$

We will now undertake the task of calculating the collision term of the Boltzmann equation in a 1. order setting. The collision terms must include scattering with other species'. The main collision process for photons is Compton scattering which schematically looks like

$$e(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e(\vec{q'}) + \gamma(\vec{p'}) \tag{2.79}$$

where $\vec{q}$ and $\vec{p}$ are the electron and photon momenta respectively. The collision term[10, 25] is

$$\begin{aligned} C[f(p)] = \frac{1}{E(p)} \int & Dq\, Dq'\, Dp'\, (2\pi)^4 \,\delta^3\left(\vec{p} + \vec{q} - \vec{p'} - \vec{q'}\right) \cdot |M|^2 \\ & \times \delta(E(\vec{p}) + E_e(\vec{q}) - E(\vec{p'}) - E(\vec{q'})) \\ & \times \left[f_e(\vec{q'}) f(\vec{p'}) - f_e(\vec{q}) f(\vec{p}) \right]. \end{aligned} \tag{2.80}$$

where $f_e(q)$ is the electron distribution function and

$$Dq = \frac{d^3q}{(2\pi)^3 2E_e(q)}. \tag{2.81}$$

is the Lorentz invariant momentum space element. The delta functions in the integral ensure energy-momentum conservation. $|M|^2$ is the Compton scattering amplitude which I will state soon. The electron energy $E(q)$ is non-relativistic, hence $E(q) = m_e + q^2/2m_e$. It is therefore correct to assume that the electrons are (classically) thermally distributed about a bulk velocity v_b. The distribution function will thus take the form[25]

$$f_e(\vec{q}) = (2\pi)^3 n_e (2\pi m_e T)^{-3/2} exp\left[\frac{-(\vec{q} - m_e\vec{v}_b)^2}{2m_e T}\right], \tag{2.82}$$

with $v_b \sim q/m_e$.
At these epochs the kinetic energy of the electron is much smaller than the rest energy, so in the division of $E(q)$ we can set $E(q) \simeq m_e$. In non-relativistic Compton scattering, very little energy is transferred between the electron and thee photon, so $E(q) - E(q')$ must be quite small. To evaluate the integral we will need to 'Taylor' expand the delta functions. We can thus expand $\delta(E(p) + E_e(q) - E(p') - E(q'))$ with respect to $(q')^2/2m_e$ about the zero order kinetic energy $(q)^2/2m_e$, remembering that delta function 'derivatives' are defined via integration by parts.
Before we expand the delta function, we can already do the q' integral, the effect of which will set $\vec{q'} = \vec{p} + \vec{q} - \vec{p'}$ in the integrands.

$$\begin{aligned} C[f(p)] = \frac{1}{p} \int \frac{d^3q}{(2\pi)^3 2m_e} \int \frac{d^3p'}{(2\pi)^3 2p'} (2\pi)^4 \frac{1}{(2\pi)^3 2m_e} \cdot |M|^2 \\ \times \delta(E(\vec{p}) + E_e(\vec{q}) - E(\vec{p'}) - E(\vec{p} + \vec{q} - \vec{p'})) \\ \times \left[f_e(\vec{p} + \vec{q} - \vec{p'}) f(\vec{p'}) - f_e(\vec{q}) f(\vec{p})\right]. \end{aligned} \tag{2.83}$$

Inserting the values of the energies leads to

$$\begin{aligned} C[f(p)] = \frac{\pi}{4m_e^2 p} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3 p'} \cdot |M|^2 \\ \times \delta\left(p + \frac{q^2}{2m_e} - p' - \frac{(\vec{q} + \vec{p} - \vec{p'})^2}{2m_e}\right) \\ \times \left[f_e(\vec{p} + \vec{q} - \vec{p'}) f(\vec{p'}) - f_e(\vec{q}) f(\vec{p})\right]. \end{aligned} \tag{2.84}$$

Since $\vec{p} - \vec{p'}$ is of the order p and $\vec{q}$ is much larger than $\vec{p}$, we can set $f_e(\vec{p} + \vec{q} - \vec{p'}) \simeq f_e(\vec{q})$ which gives

$$\begin{aligned} C[f(p)] = \frac{\pi}{4m_e^2 p} \int \frac{d^3q}{(2\pi)^3} f_e(\vec{q}) \int \frac{d^3p'}{(2\pi)^3 p'} \cdot |M|^2 \\ \times \delta\left(p + \frac{q^2}{2m_e} - p' - \frac{(\vec{q} + \vec{p} - \vec{p'})^2}{2m_e}\right) \\ \times \left[f(\vec{p'}) - f(\vec{p})\right]. \end{aligned} \tag{2.85}$$

We now need to address the task of expanding the remaining delta function. We can expand it

$$\begin{aligned}\delta\left(p+\frac{q^2}{2m_e}-p'-\frac{(\vec{q}+\vec{p}-\vec{p'})^2}{2m_e}\right) \simeq\ & \delta\left(p+E_e(q)-p'-E_e(q')\right)\Big|_{E_e(q)=E_e(q')} \\ &+ (E_e(q')-E_e(q))\times \\ &\frac{\partial}{\partial E_e(q')}\Big(\delta[p+E_e(q)-p' \\ &- E_e(q')]\Big)\Big|_{E_e(q)=E_e(q')},\end{aligned} \tag{2.86}$$

which yields

$$\begin{aligned}\delta\left(p+\frac{q^2}{2m_e}-p'-\frac{(\vec{q}+\vec{p}-\vec{p'})^2}{2m_e}\right) \simeq\ & \delta\left(p-p'\right)+(E_e(q')-E_e(q))\times \\ &\frac{\partial\delta[p+E_e(q)-p'-E_e(q')]}{\partial E_e(q')}\Big|_{E_e(q)=E_e(q')}\end{aligned}$$

We can simplify the electron energy difference $E_e(q')-E_e(q)$ by noting that since $\vec{q}$ is much larger than $\vec{p}$ and $\vec{p'}$, we have that

$$\begin{aligned}E_e(q')-E_e(q) &= \frac{(\vec{q}+\vec{p}-\vec{p'})^2}{2m_e}-\frac{q^2}{2m_e} \\ &= \frac{q^2}{2m_e}+\frac{2\,\vec{q}\cdot(\vec{p}-\vec{p'})}{2m_e}+\frac{(\vec{p}-\vec{p'})^2}{2m_e}-\frac{q^2}{2m_e} \\ &\simeq \frac{\vec{q}\cdot(\vec{p}-\vec{p'})}{m_e}.\end{aligned} \tag{2.87}$$

We now need to [10] relate the obtained delta function derivative to a derivative with respect to p'. This can be accomplished by recalling the fact that for a general function $g(x,y)=f(x-y)$ we have that $\partial g/\partial x=-\partial g/\partial y$. This can be easily seen by setting $u=x-y$ and calculating $\partial f/\partial u$ in two different ways. First with constant y

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\cdot\frac{dx}{du}=\frac{\partial f}{\partial x}, \tag{2.88}$$

and then with constant x

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial y}\cdot\frac{dy}{du}=\frac{\partial f}{\partial y}\cdot(-1)=-\frac{\partial f}{\partial y}. \tag{2.89}$$

These two expressions must be equal. Hence

$$\frac{\partial f(x-y)}{\partial x} = -\frac{\partial f(x-y)}{\partial y}. \tag{2.90}$$

We can use this result on the derivative of the delta function with $x = -E_e(q)$ and $y = p'$ which gives

$$\begin{aligned} -\frac{\partial\delta[p+E_e(q)-p'-E_e(q')]}{\partial(-E_e(q'))}\Big|_{E_e(q)=E_e(q')} &= \frac{\partial\delta[p+E_e(q)-p'-E_e(q')]}{\partial p'}\Big|_{E_e(q)=E_e(q')} \\ &= \frac{\partial\delta(p-p')}{\partial p'}. \end{aligned} \tag{2.91}$$

Using this result, the full (1.order) Taylor expansion of the delta function becomes

$$\delta\left(p+\frac{q^2}{2m_e}-p'-\frac{(\vec{q}+\vec{p}-\vec{p'})^2}{2m_e}\right) \simeq \delta\left(p-p'\right)+\frac{\vec{q}\cdot(\vec{p}-\vec{p'})}{m_e}\frac{\partial\,\delta(p-p')}{\partial p'}. \tag{2.92}$$

Inserting this result into equation (2.85) we obtain

$$\begin{aligned} C[f(p)] = \frac{\pi}{4m_e^2 p}\int\frac{d^3q}{(2\pi)^3}f_e(\vec{q})\int\frac{d^3p'}{(2\pi)^3p'}\cdot|M|^2 \\ \times\left[\delta\left(p-p'\right)+\frac{\vec{q}\cdot(\vec{p}-\vec{p'})}{m_e}\frac{\partial\,\delta(p-p')}{\partial p'}\right] \\ \times\left[f(\vec{p'})-f(\vec{p})\right]. \end{aligned} \tag{2.93}$$

Compton Scattering Amplitude

We now need to consider the Compton scattering Amplitude $|M|^2$ in the Collision integral. The Amplitude squared will in general have an angular dependence, of the order $1+\cos^2(\vec{n}\cdot\vec{n'})$, in addition to a polarization dependence, where a small fraction of CMB photons become polarized by the scattering[25]. I will in this simplified setting omit these effects. Omitting the angular part will infer a 1% error in the final results[2]. Thus in our setting, the Amplitude squared is

$$|M|^2 = 8\pi\sigma_T m_e^2, \tag{2.94}$$

[2]Adding the cosine dependence would infer an extra factor of $\tilde{\Theta}_2$ in the final results, where $\tilde{\Theta}_2$ is the quadropole moment of the temperature contrast.

where σ_T is the *Thompson cross section.* Inserted into equation (2.93) we obtain

$$C[f(p)] = \frac{2\pi^2\sigma_T}{p}\int\frac{d^3q}{(2\pi)^3}f_e(\vec{q})\int\frac{d^3p'}{(2\pi)^3p'}\left[f(\vec{p'})-f(\vec{p})\right] \times\left[\delta\left(p-p'\right)+\frac{\vec{q}\cdot(\vec{p}-\vec{p'})}{m_e}\frac{\partial\,\delta(p-p')}{\partial p'}\right]. \tag{2.95}$$

Calculating the Momentum Integrals

We are now ready to complete the task of calculating the momentum integrals in equation (2.95). For this we need to recall how to evaluate Gaussian statistical integrals, since the electron distribution function $f_e(q)$ is a normal distribution. The following integrals will thus be useful

$$\int\frac{d^3q}{(2\pi)^3}f_e(\vec{q}) = n_e \tag{2.96}$$

$$\int\frac{d^3q}{(2\pi)^3}\frac{\vec{q}}{m_e}f_e(\vec{q}) = n_e\vec{v_b}. \tag{2.97}$$

Using these results, we can go further in evaluating the collision integral (2.95).

$$C[f(p)] = \frac{2\pi^2\sigma_T}{p}\int\frac{d^3p'}{(2\pi)^3p'}\left[f(\vec{p'})-f(\vec{p})\right] \times\left[\delta\left(p-p'\right)\int\frac{d^3q}{(2\pi)^3}f_e(\vec{q}) + \int\frac{d^3q}{(2\pi)^3}f_e(\vec{q})\,\vec{q}\cdot\frac{(\vec{p}-\vec{p'})}{m_e}\frac{\partial\,\delta(p-p')}{\partial p'}\right]. \tag{2.98}$$

Using equations (2.96) and (2.97) to evaluate the q integrals we get

$$C[f(p)] = \frac{2\pi^2\sigma_T}{p}\int\frac{d^3p'}{(2\pi)^3p'}\left[f(\vec{p'})-f(\vec{p})\right] \times\left[n_e\delta\left(p-p'\right)+n_e\vec{v_b}\cdot(\vec{p}-\vec{p'})\frac{\partial\,\delta(p-p')}{\partial p'}\right]. \tag{2.99}$$

Inserting now the 1.order perturbation of the photon distribution function, equation (2.52) on page 28, we get

$$C[f(p)] = \frac{2\pi^2 n_e \sigma_T}{p} \int \frac{d^3p'}{(2\pi)^3 p'} \left[f_0(p') - f_0(p) - p' \frac{\partial f_0}{\partial p'} \Theta(\mathbf{n}') + p \frac{\partial f_0}{\partial p} \Theta(\mathbf{n}) \right] \\ \times \left[\delta\left(p - p'\right) + \vec{v_b} \cdot (\vec{p} - \vec{p'}) \frac{\partial\, \delta(p - p')}{\partial p'} \right]. \qquad (2.100)$$

When multiplying the brackets in the integral we will obtain

$$\begin{aligned} C[f(p)] = \frac{n_e \sigma_T}{p\, 4\pi} \int \frac{d^3p'}{p'} \Big[& \delta\left(p - p'\right) \left(f_0(p') - f_0(p)\right) \\ & + \delta\left(p - p'\right) \left[-p' \frac{\partial f_0}{\partial p'} \Theta(\mathbf{n}') + p \frac{\partial f_0}{\partial p} \Theta(\mathbf{n}) \right] \\ & + \vec{v_b} \cdot (\vec{p} - \vec{p'}) \frac{\partial\, \delta(p - p')}{\partial p'} \times \left(f_0(p') - f_0(p)\right) \\ & + \vec{v_b} \cdot (\vec{p} - \vec{p'}) \frac{\partial\, \delta(p - p')}{\partial p'} \\ & \quad \times \left(-p' \frac{\partial f_0}{\partial p'} \Theta(\mathbf{n}') + p \frac{\partial f_0}{\partial p} \Theta(\mathbf{n}) \right) \Big]. \end{aligned} \qquad (2.101)$$

In the above integral, we can see that the first term is 0 when integrating over p'. We can in addition neglect the last term since it contains a product of two small quantities, v_b and Θ, hence being of 2.order in nature. This simplification yields

$$C[f(p)] = \frac{n_e \sigma_T}{p\, 4\pi} \int \frac{d^3p'}{p'} \left[\delta\left(p - p'\right) \left[-p' \frac{\partial f_0}{\partial p'} \Theta(\mathbf{n}') + p \frac{\partial f_0}{\partial p} \Theta(\mathbf{n}) \right] \\ + \vec{v_b} \cdot (\vec{p} - \vec{p'}) \frac{\partial\, \delta(p - p')}{\partial p'} \left(f_0(p') - f_0(p)\right) \right]. \qquad (2.102)$$

The integral has an angular dependence through the terms $\Theta(\mathbf{n})$ and $\vec{p'}$. It will at this stage be convenient to introduce an angular integration. Recall that $\vec{p'} = p'\vec{n'}$, hence we can set $d^3p' = (p')^2 dp' d\Omega'$, where $d\Omega'$ is the solid angle element. In addition

$$\int d\Omega' = 4\pi, \qquad (2.103)$$

or specifically

$$\int d\Omega' n_i' n_j' = \frac{4\pi}{3} \delta_{ij}, \qquad (2.104)$$

and

$$\int d\Omega' n_i' = \int d\Omega' n_i' n_j' n_k' = 0. \qquad (2.105)$$

Inserting $d^3p' = (p')^2 dp' d\Omega'$ into the collision term we obtain

$$C[f(p)] = \frac{n_e \sigma_T}{p\,4\pi} \int_0^\infty dp'\,p' \int d\Omega' \Big[\delta\,(p-p') \left[-p' \frac{\partial f_0}{\partial p'} \Theta(\mathbf{n}') + p \frac{\partial f_0}{\partial p} \Theta(\mathbf{n}) \right] + \vec{v_b} \cdot (\vec{p} - \vec{p'}) \frac{\partial\,\delta(p-p')}{\partial p'} (f_0(p') - f_0(p)) \Big]. \quad (2.106)$$

For all integrands with no $\vec{n'}$ dependence, integration over solid angle yields simply a factor of 4π.

$$\begin{aligned} C[f(p)] = \frac{n_e \sigma_T}{p\,4\pi} \int_0^\infty dp'\,p' \Big[& - \delta\,(p-p')\, p' \frac{\partial f_0}{\partial p'} \int d\Omega' \Theta(\mathbf{n}') \\ & + 4\pi\,\delta\,(p-p')\, p \frac{\partial f_0}{\partial p} \Theta(\mathbf{n}) \\ & + 4\pi\,\vec{v_b} \cdot \vec{p} \frac{\partial\,\delta(p-p')}{\partial p'} (f_0(p') - f_0(p)) \\ & - \frac{\partial\,\delta(p-p')}{\partial p'} (f_0(p') - f_0(p)) \int d\Omega' \vec{v_b} \cdot \vec{p'} \Big]. \end{aligned} \quad (2.107)$$

The last term in the above equation is 0 since $\vec{v_b}$ is a fixed vector, a result from vector calculus. We now define what is considered as the *monopole* part of the perturbation $\Theta(\mathbf{n}')$

$$\Theta_0(\mathbf{x}, \eta) = \Theta_0 \equiv \frac{1}{4\pi} \int d\Omega' \Theta(\mathbf{n}'). \quad (2.108)$$

As explicitly stated above, Θ_0 is a function of position and time. It represents the deviation of the monopole at a fixed position from the average temperature in all space. With this definition the collision integral becomes

$$\begin{aligned} C[f(p)] = \frac{n_e \sigma_T}{p} \int_0^\infty dp'\,p' \Big[& - \delta\,(p-p')\, p' \frac{\partial f_0}{\partial p'} \Theta_0 + \delta\,(p-p')\, p \frac{\partial f_0}{\partial p} \Theta(\mathbf{n}) \\ & + \vec{v_b} \cdot \vec{p}\, \frac{\partial\,\delta(p-p')}{\partial p'} (f_0(p') - f_0(p)) \Big]. \end{aligned} \quad (2.109)$$

Multiplying through by the integral sign we get

$$\begin{aligned} C[f(p)] = \frac{n_e \sigma_T}{p} \Big[& - \int_0^\infty dp'\,(p')^2 \delta\,(p-p')\, \frac{\partial f_0}{\partial p'} \Theta_0 \\ & + \int_0^\infty dp'\,p' p\,\delta\,(p-p')\, \frac{\partial f_0}{\partial p} \Theta(\mathbf{n}) \\ & + \int_0^\infty dp'\,p'\,p\,\vec{v_b} \cdot \vec{n}\, \frac{\partial\,\delta(p-p')}{\partial p'} (f_0(p') - f_0(p)) \Big]. \end{aligned} \quad (2.110)$$

The first two integrals are trivially calculated to give

$$C[f(p)] = \frac{n_e \sigma_T}{p} \Big[-p^2 \frac{\partial f_0}{\partial p} \Theta_0 + p^2 \frac{\partial f_0}{\partial p} \Theta(\mathbf{n}) + \int_0^\infty dp' \, p' \, p \, \vec{v_b} \cdot \vec{n} \, \frac{\partial \, \delta(p-p')}{\partial p'} (f_0(p') - f_0(p)) \Big]. \quad (2.111)$$

As hinted on before, we solve the last integral by partial integration

$$\int_0^\infty dp' \, p' \, p \vec{v_b} \cdot \vec{n} \, \frac{\partial \, \delta(p-p')}{\partial p'} (f_0(p') - f_0(p)) = \Big[p' \, p \vec{v_b} \cdot \vec{n} \, \delta(p-p')(f_0(p') - f_0(p)) \Big]_0^\infty - \int_0^\infty dp' p \, \vec{v_b} \cdot \vec{n} \, \delta(p-p') \frac{\partial}{\partial p'} \left[p' f_0(p') - p' f_0(p) \right]. \quad (2.112)$$

The first term on the left hand side is 0. This implies

$$\int_0^\infty dp' \, p' \, p \vec{v_b} \cdot \vec{n} \, \frac{\partial \, \delta(p-p')}{\partial p'} (f_0(p') - f_0(p)) = - \int_0^\infty dp' p \, \vec{v_b} \cdot \vec{n} \, \delta(p-p') \times \left[f_0(p') + p' \frac{\partial f_0(p')}{\partial p'} - p' f_0(p) \right] \quad (2.113)$$

The right hand side of the integral is

$$\begin{aligned} \text{Integral} &= - \int_0^\infty dp' p \, p' \, \vec{v_b} \cdot \vec{n} \, \delta(p-p') \frac{\partial f_0(p')}{\partial p'} \\ &\quad - \int_0^\infty dp' \, p \, p' \, \vec{v_b} \cdot \vec{n} \, \delta(p-p')(f_0(p') - f_0(p)) \\ &= -p^2 \vec{v_b} \cdot \vec{n} \frac{\partial f_0(p)}{\partial p} - p^2 \vec{v_b} \cdot \vec{n} \, (f_0(p) - f_0(p)) \\ &= -p^2 \vec{v_b} \cdot \vec{n} \frac{\partial f_0(p)}{\partial p}. \end{aligned} \quad (2.114)$$

Inserting this into equation (2.111) results in

$$C[f(p)] = \frac{n_e \sigma_T}{p} \Big[-p^2 \frac{\partial f_0}{\partial p} \Theta_0 + p^2 \frac{\partial f_0}{\partial p} \Theta(\mathbf{n}) - p^2 \vec{v}_b \cdot \vec{n} \frac{\partial f_0(p)}{\partial p} \Big], \quad (2.115)$$

which finally gives

$$C[f(p)] = -p \, \frac{\partial f_0}{\partial p} n_e \sigma_T \big[\Theta_0 - \Theta(\mathbf{n}) + \vec{v}_b \cdot \vec{n} \big]. \quad (2.116)$$

This is the collision term of the Boltzmann equation. It quantifies the change in the distribution function as the photon interacts with the electrons through Compton scattering.

2.1.10 The Complete Photon Boltzmann Equation in Fourier Space

We can now write down the full first order Boltzmann equation for the photon with the collision term. Recall that

$$\frac{df}{d\eta} = a\,C[f] \tag{2.117}$$

Combining equations (2.66) on page 31 and (2.116) we get

$$-p\frac{\partial f_0}{\partial p}\left[\frac{\partial \Theta}{\partial \eta} + n^i\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial \eta} + n^i\frac{\partial \Psi}{\partial x^i}\right] = -p\,\frac{\partial f_0}{\partial p} a\,n_e\sigma_T\big[\Theta_0 - \Theta(\mathbf{n}) + \vec{v}_b\cdot\vec{n}\big], \tag{2.118}$$

which leads to

$$\frac{\partial \Theta}{\partial \eta} + n^i\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial \eta} + n^i\frac{\partial \Psi}{\partial x^i} = a\,n_e\sigma_T\big[\Theta_0 - \Theta(\mathbf{n}) + \vec{v}_b\cdot\vec{n}\big]. \tag{2.119}$$

Going now to Fourier space we get

$$\dot{\tilde{\Theta}} + i\,k\mu\,\tilde{\Theta} + \dot{\tilde{\Phi}} + i\,k\mu\,\tilde{\Psi} = a\,n_e\sigma_T\big[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\,\tilde{v}_b\big]. \tag{2.120}$$

where we have assumed that the baryonic bulk velocity $\vec{v}_b$ is irrotational ($\nabla\times\vec{v}_b = 0$), which implies that it has the same direction as $\vec{k}$ leading to $\vec{\tilde{v}}_b\cdot\vec{n} = \tilde{v}_b\mu$. It is now convenient to introduce the *optical depth*[25] τ defined as

$$\tau(\eta) \equiv \int_\eta^{\eta_0} d\eta' a\,n_e\sigma_T, \tag{2.121}$$

which is related to the mean free path of the photon. Differentiating this equation with respect to conformal time gives

$$\dot{\tau} = -a\,n_e\sigma_T, \tag{2.122}$$

which when inserted into equation (2.120) gives

$$\dot{\tilde{\Theta}} + i\,k\mu\,\tilde{\Theta} + \dot{\tilde{\Phi}} + i\,k\mu\,\tilde{\Psi} = -\dot{\tau}\big[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\,\tilde{v}_b\big]. \tag{2.123}$$

This is the sought after Boltzmann equation of the photon. It is the first of the fundamental equations we will need to study the cosmic plasma.

2.2 The Boltzmann Equation for Baryonic Matter

We will now in this section find the Boltzmann equation for the baryonic species' of the Universe, mainly the electrons and the protons. In cosmology it has been common practice to refer to the electrons as baryons although they are leptons. I will perpetuate this 'abuse' of terminology in this work.
I will specifically need an equation for the baryon velocity v_b introduced in section 2.1.9 to be able to calculate the photon power spectrum, since the Boltzmann equation of the photon explicitly depends on v_b. We will thus need to run through all the calculations as we did for the photon, but now in a slightly different setting. For instance, the species' we are studying now are massive and a few more interactions must be included in the collision term. But let us now proceed with the calculations, starting with the geodesic equation of a massive particle.

2.2.1 The Geodesic Equation of a Massive Particle

Recall that the geodesic equation for a free particle is

$$\frac{dP^\mu}{d\eta} = g^{\mu\nu}\left(\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\nu} - \frac{\partial g_{\nu\alpha}}{\partial x^\beta}\right)\frac{P^\alpha P^\beta}{P^0}. \tag{2.124}$$

For a particle with mass m and energy E we have that

$$\begin{aligned} P^2 &\equiv g_{\mu\nu}P^\mu P^\nu = -m^2 \\ E &= \sqrt{p^2+m^2}. \end{aligned} \tag{2.125}$$

To find an expression for the 4-momentum P^μ, we define again $p^2 = g_{ij}P^iP^j$ which gives

$$\begin{aligned} g_{00}(P^0)^2 + g_{ij}P^iP^j &= -m^2 \\ -a^2(1+2\Psi)(P^0)^2 + p^2 &= -m^2 \\ (P^0)^2 &= \frac{p^2+m^2}{a^2(1+2\Psi)} \\ P^0 &= \frac{E}{a\sqrt{1+2\Psi}} \approx E\frac{1-\Psi}{a}. \end{aligned} \tag{2.126}$$

In finding the spatial components P^i we follow the same line of calculation as in section 2.1.2 for the photon. Hence the 4-momentum is

$$P^\mu = \left[E\frac{1-\Psi}{a}, p\,n^i\frac{1-\Phi}{a}\right]. \tag{2.127}$$

We will need the expression for $dx^i/d\eta$ in the Boltzmann Equation, this is simply

$$\frac{dx^i}{d\eta} = \frac{P^i}{P^0} = \frac{p\,n^i\frac{1-\Phi}{a}}{E\frac{1-\Psi}{a}} \approx \frac{p}{E}n^i(1-\Phi+\Psi). \tag{2.128}$$

Some other useful expressions are

$$\delta_{ij}P^iP^j = \delta_{ij}\,p^2\,n^in^j\frac{(1-\Phi)^2}{a^2} \approx \frac{p^2}{a^2}(1+2\Phi), \tag{2.129}$$

in addition

$$\delta_{ij}\frac{P^iP^j}{P^0} = \frac{\frac{p^2}{a^2}(1+2\Phi)}{\frac{E}{a}(1-\Psi)} \approx \frac{p^2}{aE}(1-2\Phi+\Psi). \tag{2.130}$$

We will again only need the zeroth component of the geodesic equation

$$\frac{dP^0}{d\eta} = g^{0\nu}\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\frac{P^\alpha P^\beta}{P^0} - g^{0\nu}\frac{\partial g_{\nu\alpha}}{\partial x^\beta}\frac{P^\alpha P^\beta}{P^0}. \tag{2.131}$$

Starting with the left hand side of the above equation, we get

$$\begin{aligned}
\frac{dP^0}{d\eta} &= \frac{d}{d\eta}\left(E(1-\Psi)\frac{1}{a}\right) \\
&= \frac{1}{a}\frac{dE}{d\eta}(1-\Psi) - \frac{E}{a}\frac{d\Psi}{d\eta} - E(1-\Psi)\mathcal{H} \\
&= \frac{1}{a}\frac{dE}{d\eta}(1-\Psi) - \frac{E}{a}\left(\frac{\partial\Psi}{\partial\eta} + \frac{\partial\Psi}{\partial x^i}\frac{dx^i}{d\eta}\right) - E(1-\Psi)\mathcal{H} \\
&= \frac{1}{a}\frac{dE}{d\eta}(1-\Psi) - \frac{E}{a}\frac{\partial\Psi}{\partial\eta} - \frac{p}{a}n^i\frac{\partial\Psi}{\partial x^i} - E(1-\Psi)\mathcal{H},
\end{aligned} \tag{2.132}$$

where in the last line I have used equation (2.128) and omitted the second order terms.
Working now with the right hand side of equation (2.131), the first term is

$$\begin{aligned}
\frac{1}{2}g^{0\nu}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\frac{P^\alpha P^\beta}{P^0} &= \frac{1}{2}g^{00}\frac{\partial g_{\alpha\beta}}{\partial x^0}\frac{P^\alpha P^\beta}{P^0} \\
&= \frac{1}{2}g^{00}g_{00,0}P^0 + \frac{1}{2}g^{00}g_{ij,0}\frac{P^iP^j}{P^0} \\
&= \frac{\dot a}{a}P^0 + \frac{\partial\Psi}{\partial\eta}P^0 \\
&\quad - \left(\frac{\dot a}{a}(1-2\Psi+2\Phi) + \frac{\partial\Phi}{\partial\eta}\right)\delta_{ij}\frac{P^iP^j}{P^0}.
\end{aligned} \tag{2.133}$$

Using equations (2.129) and (2.130) we obtain

$$\frac{1}{2}g^{0\nu}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\frac{P^\alpha P^\beta}{P^0} = \mathcal{H}\frac{E}{a}(1-\Psi) + \frac{E}{a}\frac{\partial\Psi}{\partial\eta} - \mathcal{H}\frac{p^2}{aE}(1-\Psi) - \frac{p^2}{aE}\frac{\partial\Phi}{\partial\eta}. \quad (2.134)$$

where we have again omitted all 2.order terms in the calculations. For the second term of equation (2.131), the calculations are identical to the ones in section 2.1.2 on page 25. Hence we obtain

$$\begin{aligned} g^{0\nu}\frac{\partial g_{\nu\alpha}}{\partial x^\beta}\frac{P^\alpha P^\beta}{P^0} &= 2\mathcal{H}P^0 + 2\frac{\partial\Psi}{\partial\eta}P^0 + 2\frac{\partial\Psi}{\partial x^i}P^i \\ &= 2\mathcal{H}\frac{E}{a}(1-\Psi) + 2\frac{E}{a}\frac{\partial\Psi}{\partial\eta} + 2\frac{\partial\Psi}{\partial x^i}\frac{p}{a}n^i. \end{aligned} \quad (2.135)$$

Subtracting equations (2.134) from (2.135), we obtain the left hand side of the geodesic equation

$$-\mathcal{H}\frac{E}{a}(1-\Psi) - \mathcal{H}\frac{p^2}{aE}(1-\Psi) - \frac{E}{a}\frac{\partial\Psi}{\partial\eta} - \frac{p^2}{aE}\frac{\partial\Phi}{\partial\eta} - 2\frac{\partial\Psi}{\partial x^i}\frac{p}{a}n^i. \quad (2.136)$$

Combining equations (2.136) and (2.132), we get after some rearrangement

$$\frac{dE}{d\eta} = -\mathcal{H}\frac{p^2}{E} - \frac{p^2}{E}\frac{\partial\Phi}{\partial\eta} - \frac{\partial\Psi}{\partial x^i}p\,n^i. \quad (2.137)$$

This is the geodesic equation of a free particle with non-zero mass. Observe how similar it is to equation (2.40), the massless geodesic equation. We obviously regain the massless equation by setting $E = p$ in the above equation.

2.2.2 Baryonic Boltzmann Equation

We are now ready to write down the Boltzmann equation for the electrons and protons. Each species will have an equation describing the change in their distribution function f through phase space. The left hand side of the Boltzmann equation can be expanded as

$$\frac{df}{d\eta} = \frac{\partial f}{\partial\eta} + \frac{dx^i}{d\eta}\frac{\partial f}{\partial x^i} + \frac{dE}{d\eta}\frac{\partial f}{\partial E} + \frac{dn^i}{d\eta}\frac{\partial f}{\partial n^i}. \quad (2.138)$$

The last term can be ignored by the same argument as in section 2.1.3. Using equation (2.137), the geodesic equation, and equation (2.128) for the velocity, we obtain

$$\frac{df}{d\eta} = \frac{\partial f}{\partial\eta} + n^i\frac{p}{E}\frac{\partial f}{\partial x^i} - \frac{\partial f}{\partial E}\left[\mathcal{H}\frac{p^2}{E} + \frac{p^2}{E}\frac{\partial\Phi}{\partial\eta} + \frac{\partial\Psi}{\partial x^i}p\,n^i\right]. \quad (2.139)$$

Hence for the two species' we are dealing with, we have two sets of equations taking on the form

$$\frac{d\, f_e(\vec{x}, \vec{q}, \eta)}{d\eta} = a\, C_e[f_e] \tag{2.140}$$

$$\frac{d\, f_p(\vec{x}, \vec{Q}, \eta)}{d\eta} = a\, C_p[f_p], \tag{2.141}$$

where we have defined the electron and proton momentum as $\vec{q}$ and $\vec{Q}$ respectively. The electron collision term $C_e[f_e]$ will include Compton scattering in addition to a Coulomb scattering term with the protons. The proton collision term $C_p[f_p]$ will only include Coulomb interaction with electrons, hence neglecting scattering with photons. The reason for this is that the cross section for this process is much smaller than the Coulomb cross section[25], so we can ignore the former completely. The tight coupling between the electrons and the protons implies an equality of the overdensities of the respective species

$$\delta_b \equiv \frac{\rho_e - \rho_e^0}{\rho_e^0} = \frac{\rho_p - \rho_p^0}{\rho_p^0}, \tag{2.142}$$

where ρ_e^0 and ρ_p^0 are the zeroth order densities. We also have that the velocities of the two species' are equal, i.e.

$$\vec{v}_e = \vec{v}_p \equiv \vec{v}_b. \tag{2.143}$$

The velocity $\vec{v}_b$ is the baryon bulk velocity introduced earlier. We will now undertake the task of finding the governing equations for δ_b and $\vec{v}_b$.

2.2.3 Moments of the Boltzmann Equation

Recall that for the photon we proceeded at this stage of the calculations to obtain an expression for the perturbed distribution function f and used this in conjunction with the Boltzmann equation. We will not do this for the baryons. Instead we will take the *moment* of the Boltzmann equation, specifically the zeroth moment. Let us first recall some important integrals involving the distribution function. Firstly, the number density n is given by

$$n = \int \frac{d^3q}{(2\pi)^3} f \tag{2.144}$$

and for the velocity v^i average

$$v^i = \frac{1}{n} \int \frac{d^3q}{(2\pi)^3} f\, \frac{q}{E}\, n^i. \tag{2.145}$$

Observe then that the momentum average mv^i is simply equation (2.145) but without the division of E. Working now with the left hand side of the electron Boltzmann equation, taking the zeroth moment consists of multiplying the Boltzmann equation with $\frac{d^3q}{(2\pi)^3}$ and integrate, recalling to do the same with the collision term later. Using equation (2.139) with $p = q$ we obtain

$$\begin{aligned}\int \frac{d^3q}{(2\pi)^3}\frac{df_e}{d\eta} = \frac{\partial}{d\eta}\int \frac{d^3q}{(2\pi)^3} f_e + \frac{\partial}{\partial x^i}\int \frac{d^3q}{(2\pi)^3} f_e \frac{q}{E_e} n^i \\ - \left[\mathcal{H} + \frac{\partial\Phi}{\partial\eta}\right]\int \frac{d^3q}{(2\pi)^3}\frac{\partial f_e}{\partial E_e}\frac{q^2}{E_e} \\ - \frac{\partial\Psi}{\partial x^i}\int \frac{d^3q}{(2\pi)^3}\frac{\partial f_e}{\partial E_e} n^i q.\end{aligned} \tag{2.146}$$

Using equations (2.144) and (2.145), the first two terms are trivial. We get

$$\begin{aligned}\int \frac{d^3q}{(2\pi)^3}\frac{df_e}{d\eta} = \frac{\partial\, n_e}{d\eta} + \frac{\partial(n_e v_b^i)}{d\, x^i} - \left[\mathcal{H} + \frac{\partial\Phi}{\partial\eta}\right]\int \frac{d^3q}{(2\pi)^3}\frac{\partial f_e}{\partial E_e}\frac{q^2}{E_e} \\ - \frac{\partial\Psi}{\partial x^i}\int \frac{d^3q}{(2\pi)^3}\frac{\partial f_e}{\partial E_e} n^i q.\end{aligned} \tag{2.147}$$

The last term is at least of a 2. order nature, since the integral is only non-zero for the perturbed part of f, and is then multiplied by the $\frac{\partial\Psi}{\partial x^i}$ which is at least of 1. order. We can hence neglect this term. Before we move on to the third term, recall that since $E^2 = q^2 + m_e^2$, we have that

$$2E\, dE = 2q\, dq, \tag{2.148}$$

or

$$dE = \frac{q}{E} dq. \tag{2.149}$$

This gives that $\frac{\partial}{\partial E} = \frac{E}{q}\frac{\partial}{\partial q}$ which we will use to interchange the E derivative with a q derivative. Working now with the integral in the third term we get

$$\begin{aligned}\int \frac{d^3q}{(2\pi)^3}\frac{q^2}{E_e}\frac{\partial}{\partial E_e} f_e &= \int \frac{d^3q}{(2\pi)^3} q \frac{\partial f_e}{\partial q} \\ &= \frac{4\pi}{(2\pi)^3}\int_0^\infty dq\, q^3 \frac{\partial f_e}{\partial q} \\ &= \frac{4\pi}{(2\pi)^3}\left[\frac{1}{4} q^4 f_e\right]_0^\infty - \frac{4\pi}{(2\pi)^3} 3 \int_0^\infty dq\, q^2 f_e.\end{aligned} \tag{2.150}$$

The first term is 0 since $f \to 0$ exponentially as $q \to \infty$. Hence

$$\begin{aligned}\int \frac{d^3q}{(2\pi)^3}\frac{q^2}{E_e}\frac{\partial}{\partial E_e} f_e &= -3\,\frac{4\pi}{(2\pi)^3}\int_0^\infty dq\, q^2 f_e \\ &= -3\int \frac{d^3q}{(2\pi)^3} f_e \\ &= -3\, n_e. \end{aligned} \tag{2.151}$$

Thus the zeroth moment of the Boltzmann equation becomes

$$\int \frac{d^3q}{(2\pi)^3}\frac{df_e}{d\eta} = \frac{\partial\, n_e}{d\eta} + \frac{\partial(n_e v_b^i)}{\partial x^i} + 3\left[\mathcal{H} + \frac{\partial\Phi}{\partial\eta}\right] n_e. \tag{2.152}$$

This equation looks very much like the continuity equation without a source term, except for the last term which accounts for an expanding universe. Note that this equation is equally valid for the protons.

2.2.4 Zeroth Order Moment Equation

As we did for the photon, let us now see what information we can extract from a zeroth order expansion of the moment equation we found in the previous section. To zeroth order in equation (2.152), the velocity term v_b^i and the term including the Φ derivative will vanish. As for the photon, the 0't order Collision term is also 0. We are left with

$$\frac{\partial\, n_e^0}{d\eta} + 3\mathcal{H} n_e^0 = 0 \tag{2.153}$$

which gives

$$\frac{\partial\, n_e^0}{d\eta} = -3\frac{\dot{a}}{a} n_e^0. \tag{2.154}$$

Integrating gives

$$\ln(n_e^0) \propto -3\ln a \tag{2.155}$$

This implies that $n_e^0 \propto a^{-3}$. Hence the density of the electrons, or the baryons in general, scales as a^{-3}. Again a result previously anticipated in chapter 1.

2.2.5 The Baryon Matter Density Equation

Going now to first order, we set $n_e = n_e^0(1+\delta_b)$ which is the first order perturbation to the number density. Using equation (2.152) we get

$$\begin{aligned}
\int \frac{d^3q}{(2\pi)^3}\frac{d\,f_e}{d\eta} &= \frac{\partial}{d\eta}\left(n_e^0(1+\delta_b)\right) + \frac{\partial}{\partial x^i}\left(n_e^0 v_b^i(1+\delta_b)\right) \\
&\quad + 3\left[\mathcal{H} + \frac{\partial\Phi}{\partial\eta}\right]\left(n_e^0(1+\delta_b)\right) \\
&= \frac{\partial n_e^0}{d\eta}(1+\delta_b) + n_e^0\frac{\partial\delta_b}{d\eta} + \frac{\partial(n_e^0 v_b^i)}{\partial x^i}(1+\delta_b) \\
&\quad + \frac{\partial\delta_b}{\partial x^i}n_e^0 v_b^i + 3\mathcal{H}n_e^0(1+\delta_b) + 3\frac{\partial\Phi}{\partial\eta}n_e^0(1+\delta_b). \qquad (2.156)
\end{aligned}$$

The first and the fifth term cancel, as seen from equation (2.154). In addition we will neglect all terms involving products of v_b^i and δ_b, or $\frac{\partial\Phi}{\partial\eta}$ and δ_b, since these are of a higher order. And since n_e^0 is independent of x^i we obtain

$$\int \frac{d^3q}{(2\pi)^3}\frac{d\,f_e}{d\eta} = n_e^0\frac{\partial\delta_b}{d\eta} + n_e^0\frac{\partial v_b^i}{\partial x^i} + 3\frac{\partial\Phi}{\partial\eta}n_e^0. \qquad (2.157)$$

For the full equation we will also need the collision term which we will handle in section 2.2.7 .

2.2.6 1. Moment Of The Boltzmann Equation

In the previous section we found that integrating the Boltzmann equation over q-space gave us eventually an equation for the matter density perturbation δ_b. It will soon be clear that by taking the 1. moment of the Boltzmann equation, we will obtain (eventually) an equation for the baryon velocity v_b. Taking the first moment consists of multiplying by $\frac{d^3q}{(2\pi)^3}\vec{q}$ and integrating. The first moment of equation (2.139) is then

$$\begin{aligned}
\int \frac{d^3q}{(2\pi)^3}\vec{q}\frac{df_e}{d\eta} &= \frac{\partial}{\partial\eta}\int\frac{d^3q}{(2\pi)^3}f_e q\, n^j + \frac{\partial}{\partial\, x^i}\int\frac{d^3q}{(2\pi)^3}q\, n^j f_e\frac{q}{E_e}\, n^i \\
&\quad - \left[\mathcal{H} + \frac{\partial\Phi}{\partial\eta}\right]\int\frac{d^3q}{(2\pi)^3}q\, n^j\frac{\partial f_e}{\partial E_e}\frac{q^2}{E_e} \\
&\quad - \frac{\partial\Psi}{\partial x^i}\int\frac{d^3q}{(2\pi)^3}q\, n^j\frac{\partial f_e}{\partial E_e}n^i q, \qquad (2.158)
\end{aligned}$$

which gives

$$\begin{aligned}\int \frac{d^3q}{(2\pi)^3}\vec{q}\,\frac{df_e}{d\eta} &= \frac{\partial}{\partial\eta}\int \frac{d^3q}{(2\pi)^3} f_e q\, n^j + \frac{\partial}{\partial\, x^i}\int \frac{d^3q}{(2\pi)^3} f_e \frac{q^2}{E_e}\, n^i n^j \\ &\quad - \left[\mathcal{H} + \frac{\partial\Phi}{\partial\eta}\right]\int \frac{d^3q}{(2\pi)^3}\frac{\partial f_e}{\partial E_e}\frac{q^3}{E_e} n^j \\ &\quad - \frac{\partial\Psi}{\partial x^i}\int \frac{d^3q}{(2\pi)^3}\frac{\partial f_e}{\partial E_e} q^2 n^i n^j . \end{aligned} \tag{2.159}$$

We can neglect the second term since it is of the order $\frac{q^2}{E_e}$. The integral in the first term is simply $m_e n_e v_b^j$. In the remaining terms we introduce again $\frac{q}{E}\frac{\partial}{\partial E} = \frac{\partial}{\partial q}$. This gives

$$\begin{aligned}\int \frac{d^3q}{(2\pi)^3}\vec{q}\,\frac{df_e}{d\eta} &= m_e \frac{\partial n_e v_b^j}{\partial\eta} - \left[\mathcal{H} + \frac{\partial\Phi}{\partial\eta}\right]\int \frac{d^3q}{(2\pi)^3}\frac{\partial f_e}{\partial q} q^2 n^j \\ &\quad - \frac{\partial\Psi}{\partial x^i}\int \frac{d^3q}{(2\pi)^3}\frac{\partial f_e}{\partial q} q\, E_e n^i n^j . \end{aligned} \tag{2.160}$$

Working now with the integral in the second term, we get

$$\begin{aligned}\int \frac{d^3q}{(2\pi)^3}\frac{\partial f_e}{\partial q} q^2 n^j &= \int \frac{d\Omega\, n^j}{(2\pi)^3}\int_0^\infty dq\, q^4 \frac{\partial f_e}{\partial q} \\ &= \int \frac{d\Omega\, n^j}{(2\pi)^3}\left[q^4 f_e\right]_0^\infty - 4\int \frac{d\Omega\, n^j}{(2\pi)^3}\int_0^\infty dq\, q^3 f_e \\ &= -4\int \frac{d\Omega\, n^j}{(2\pi)^3}\int_0^\infty dq\, q^3 f_e \\ &= -4\int \frac{d^3q}{(2\pi)^3} f_e q\, n^j \\ &= -4\, m_e n_e v_b^j . \end{aligned} \tag{2.161}$$

For the integral in the last term in equation (2.160), we get

$$\begin{aligned}\int \frac{d^3q}{(2\pi)^3}\frac{\partial f_e}{\partial q} q\, E_e n^i n^j &= \int \frac{d\Omega}{(2\pi)^3} n^i n^j \int_0^\infty dq\, \frac{\partial f_e}{\partial q} q^3\, E_e \\ &= \int \frac{d\Omega}{(2\pi)^3} n^i n^j \left(\left[q^3 E_e\, f_e\right]_0^\infty - \int_0^\infty dq\, f_e \frac{\partial(q^3 E_e)}{\partial q}\right) \\ &= -\int \frac{d\Omega}{(2\pi)^3} n^i n^j \int_0^\infty dq\, f_e\left[3q^2 E_e + \frac{q^4}{E_e}\right] . \end{aligned} \tag{2.162}$$

The q^4 term is completely negligible. Using the fact that in the solid angle integral we can freely set $n^i n^j = \delta^{ij}/3$, we get

$$\begin{aligned}
\int \frac{d^3q}{(2\pi)^3}\frac{\partial f_e}{\partial q} q\, E_e n^i n^j &= -\int \frac{d\Omega}{(2\pi)^3}\delta^{ij}\int_0^\infty dq\, f_e q^2 E_e \\
&= -\delta^{ij}\int \frac{d^3q}{(2\pi)^3} f_e E_e \\
&= -\delta^{ij} m_e n_e.
\end{aligned} \tag{2.163}$$

Using these obtained results with equation (2.160), we obtain

$$\int \frac{d^3q}{(2\pi)^3}\vec{q}\,\frac{df_e}{d\eta} = m_e\frac{\partial(n_e v_b^j)}{\partial\eta} - \left[\mathcal{H} + \frac{\partial\Phi}{\partial\eta}\right](-4\, m_e n_e v_b^j) - \frac{\partial\Psi}{\partial x^i}(-\delta^{ij} m_e n_e). \tag{2.164}$$

Omitting the $\frac{\partial\Phi}{\partial\eta}$ term since it multiplies the first order term v_b^j and n_e, we finally obtain the 1. moment of the Boltzmann equation

$$\int \frac{d^3q}{(2\pi)^3}\vec{q}\,\frac{df_e}{d\eta} = m_e\frac{\partial(n_e v_b^j)}{\partial\eta} + 4\mathcal{H} m_e n_e v_b^j + \frac{\partial\Psi}{\partial x^j} m_e n_e. \tag{2.165}$$

Note that this equation is completely of 2. order. Hence we need to linearize it. We can do this by exchanging the perturbed number density n_e with the zero order density n_e^0. This gives

$$\begin{aligned}
\int \frac{d^3q}{(2\pi)^3}\vec{q}\,\frac{df_e}{d\eta} &= m_e\frac{\partial(n_e^0 v_b^j)}{\partial\eta} + 4\mathcal{H} m_e n_e^0 v_b^j + \frac{\partial\Psi}{\partial x^j} m_e n_e^0 \\
&= m_e\frac{\partial n_e^0}{\partial\eta} v_b^j + m_e n_e^0\frac{\partial v_b^j}{\partial\eta} + 4\mathcal{H} m_e n_e^0 v_b^j + \frac{\partial\Psi}{\partial x^j} m_e n_e^0 \\
&= m_e n_e^0\frac{\partial v_b^j}{\partial\eta} + \mathcal{H} m_e n_e^0 v_b^j + \frac{\partial\Psi}{\partial x^j} m_e n_e^0 \\
&= \rho_e\frac{\partial v_b^j}{\partial\eta} + \mathcal{H}\rho_e v_b^j + \frac{\partial\Psi}{\partial x^j}\rho_e.
\end{aligned} \tag{2.166}$$

Where I have used that $\partial n_e^0/\partial\eta = -3\mathcal{H} n_e^0$ and set $\rho_e = m_e n_e^0$. This equation is equally valid for the protons, giving an equation of the form

$$\int \frac{d^3Q}{(2\pi)^3}\vec{Q}\,\frac{df_p}{d\eta} = \rho_p\frac{\partial v_b^j}{\partial\eta} + \mathcal{H}\rho_p v_b^j + \frac{\partial\Psi}{\partial x^j}\rho_p. \tag{2.167}$$

To go further now, we will need to address the collision terms of the Boltzmann equations and their moments.

2.2.7 The Collision Terms Of The Baryons

We now need to work with the collision terms of the Boltzmann equation of the baryons. It will now be convenient to introduce a functional notation from [10], for the involved integrals by

$$< X >_{pp'q'} \equiv \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} X. \tag{2.168}$$

Hence the collision term of the electron-photon interaction can be written in this notation simply by $< C_{e\gamma} >_{pp'q'}$ where

$$\begin{aligned} C_{e\gamma} &= (2\pi)^4 \, \delta^3 \left(\vec{p} + \vec{q} - \vec{p'} - \vec{q'} \right) \cdot \frac{|M|^2}{E(p)E(p')E(q)E(q')} \\ &\times \delta(E(\vec{p}) + E_e(\vec{q}) - E(\vec{p'}) - E(\vec{q'})) \\ &\times \left[f_e(\vec{q'}) f(\vec{p'}) - f_e(\vec{q}) f(\vec{p}) \right]. \end{aligned} \tag{2.169}$$

We can thus write the unintegrated Boltzmann equations for the electrons and the protons as

$$\frac{1}{a}\frac{df_e(q)}{d\eta} =< C_{ep} >_{QQ'q'} + < C_{e\gamma} >_{pp'q'} \tag{2.170}$$

$$\frac{1}{a}\frac{df_p(Q)}{d\eta} =< C_{ep} >_{qq'Q'}, \tag{2.171}$$

where the integrand C_{ep} is the Coulomb interaction term which otherwise has the same form as $C_{e\gamma}$.

Complete Density Perturbation Equation

If we now study the zeroth moment equation again, recall that we obtained in section 2.2.5 the right hand side of the zeroth moment of equation (2.170). Using equation (2.157) we get

$$n_e^0 \frac{\partial \delta_b}{d\eta} + n_e^0 \frac{\partial v_b^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial \eta} n_e^0 = a < C_{ep} >_{QQ'q'q} + a < C_{e\gamma} >_{pp'q'q} . \tag{2.172}$$

Observe now that the integration measure of both collision terms are symmetric under the interchange $Q \to Q'$ and $q \to q'$, and $p \to p'$ and $q \to q'$, respectively[10]. The integrands C_{ep} and $C_{e\gamma}$ are both antisymmetric under this interchange. Hence the collision integrals must both be 0. Thus the zeroth moment of the Boltzmann equation is

$$n_e^0 \frac{\partial \delta_b}{d\eta} + n_e^0 \frac{\partial v_b^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial \eta} n_e^0 = 0. \tag{2.173}$$

dividing by n_e^0 finally gives

$$\frac{\partial \delta_b}{d\eta} + \frac{\partial v_b^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial \eta} = 0. \tag{2.174}$$

We have finally obtained the density perturbation equation of the baryons.

Complete 1. Moment Equation

Going now to the first moment of the Boltzmann equation, recall that we multiplied by $\frac{d^3q}{(2\pi)^3}\vec{q}$ in the electron equation and integrated. Equivalently we multiply with $\frac{d^3Q}{(2\pi)^3}\vec{Q}$ in the proton Boltzmann equation. Using equations (2.166) and (2.167) we get

$$\begin{aligned}
m_p n_p^0 \frac{\partial v_b^j}{\partial \eta} + \mathcal{H} m_p n_p^0 v_b^j + \frac{\partial \Psi}{\partial x^j} m_p n_p^0 &= a < C_{ep}\, Q^j >_{qq'Q'Q} \\
m_e n_e^0 \frac{\partial v_b^j}{\partial \eta} + \mathcal{H} m_e n_e^0 v_b^j + \frac{\partial \Psi}{\partial x^j} m_e n_e^0 &= a < C_{ep}\, q^j >_{QQ'q'q} + a < C_{e\gamma}\, q^j >_{pp'q'q} .
\end{aligned}$$

Recall that $m_p \gg m_e$, thus by adding these two equations renders the electron terms on the left hand side negligible. We then get

$$\begin{aligned}
m_p n_p^0 \frac{\partial v_b^j}{\partial \eta} + \mathcal{H} m_p n_p^0 v_b^j + \frac{\partial \Psi}{\partial x^j} m_p n_p^0 &= a < C_{ep}\, (Q^j + q^j) >_{QQ'q'q} \\
&\quad + a < C_{e\gamma}\, q^j >_{pp'q'q} .
\end{aligned} \tag{2.175}$$

The first term on the right hand side is 0 because of the antisymmetry of the integrand. Introducing the density ρ_b we finally obtain

$$\rho_b \frac{\partial v_b^j}{\partial \eta} + \mathcal{H} \rho_b v_b^j + \frac{\partial \Psi}{\partial x^j} \rho_b = a < C_{e\gamma}\, q^j >_{pp'q'q} . \tag{2.176}$$

2.2.8 Calculation of $< C_{e\gamma}\, q^j >_{pp'q'q}$

Since $q^j + p^j$ is a conserved quantity, where p^j is the photon momentum, we must have that

$$< C_{e\gamma}\, (q^j + p^j) >_{pp'q'q} = 0, \tag{2.177}$$

since $q^j + p^j = q'^j + p'^j$. Hence

$$< C_{e\gamma}\, q^j >_{pp'q'q} = - < C_{e\gamma}\, p^j >_{pp'q'q} . \tag{2.178}$$

Recall that we have calculated $< C_{e\gamma} >_{p'q'q}$ before, which is nothing else than the photon collision term. We now only need to multiply this expression by

$\frac{d^3p}{(2\pi)^3}p^j$ and integrate. It will now be convenient to go to Fourier space for the remaining calculations. In Fourier space equation (2.176) becomes

$$\rho_b\dot{\tilde{v}}_b^j + \mathcal{H}\rho_b\tilde{v}_b^j + i\,k^j\tilde{\Psi}\rho_b = -a < C_{e\gamma}\,p^j >_{pp'q'q}\,. \tag{2.179}$$

We have earlier assumed that the velocity field $\tilde{v}_b^j$ is irrotational , hence we can write $\tilde{v}_b^j = \tilde{v}_b k^j/k$. If we now contract with k^j in the above equation we get

$$\rho_b\dot{\tilde{v}}_b k + \mathcal{H}\rho_b\tilde{v}_b k + i\,k^2\tilde{\Psi}\rho_b = -a < C_{e\gamma}\,k\mu p >_{pp'q'q}, \tag{2.180}$$

or simply

$$\dot{\tilde{v}}_b + \mathcal{H}\tilde{v}_b + i\,k\tilde{\Psi} = -\frac{a}{\rho_b} < C_{e\gamma}\,\mu p >_{pp'q'q}, \tag{2.181}$$

where we have used that $\vec{k}\cdot\vec{p} = k\mu p$. Using the Fourier space version of equation (2.116) on page 40, the right hand side becomes

$$\begin{aligned}
-\frac{a}{\rho_b} < C_{e\gamma}\,\mu p >_{pp'q'q} &= -\frac{a}{\rho_b}\int \frac{d^3p}{(2\pi)^3}\mu\,p < C_{e\gamma} >_{p'q'q} \\
&= \frac{a}{\rho_b}\int \frac{d^3p}{(2\pi)^3}\mu\,p^2\,\frac{\partial f_0}{\partial p}n_e\sigma_T\big[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\,\tilde{v}_b\big] \\
&= \frac{an_e\sigma_T}{\rho_b}\int \frac{d^3p}{(2\pi)^3}\mu\,p^2\,\frac{\partial f_0}{\partial p}\big[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\,\tilde{v}_b\big].
\end{aligned} \tag{2.182}$$

Using now that $\int d\Omega = 4\pi\int_{-1}^{1} d\mu/2$ we get

$$-\frac{a}{\rho_b} < C_{e\gamma}\,\mu p >_{pp'q'q} = \frac{an_e\sigma_T}{\rho_b}\int_0^\infty \frac{dp}{2\pi^2}p^4\,\frac{\partial f_0}{\partial p}\int_{-1}^{1}\frac{d\mu}{2}\big[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\,\tilde{v}_b\big]. \tag{2.183}$$

If we calculate the p integral first, we obtain

$$\begin{aligned}
\int_0^\infty \frac{dp}{2\pi^2}p^4\,\frac{\partial f_0}{\partial p} &= \left[\frac{p^4}{2\pi^2}f_0\right]_0^\infty - 4\int_0^\infty \frac{dp}{2\pi^2}p^3 f_0 \\
&= -4\int_0^\infty \frac{dp}{2\pi^2}p^3 f_0.
\end{aligned} \tag{2.184}$$

Inserting the expression for the distribution function we get

$$\begin{aligned}\int_0^\infty \frac{dp}{2\pi^2} p^4 \frac{\partial f_0}{\partial p} &= -\frac{4}{2\pi^2}\int_0^\infty dp \frac{2\,p^3}{e^{p/T}-1} \\ &= -\frac{4}{2\pi^2} \times \frac{B_4}{8}\left(\frac{2\pi}{1/T}\right)^4 \\ &= -\frac{4}{2\pi^2} \times \frac{1/30}{8}\,(2\pi)^4\,T^4 \\ &= -4\,\frac{\pi^2}{15} T^4 \\ &= -4\,\rho_\gamma, \end{aligned} \tag{2.185}$$

where B_4 is the fourth Bernoulli number[1]. Returning now to the μ integral we get

$$\begin{aligned}\int_{-1}^1 \frac{d\mu}{2}\mu\big[\tilde\Theta_0 - \tilde\Theta + \mu\,\tilde v_b\big] &= \int_{-1}^1 \frac{d\mu}{2}\mu\tilde\Theta_0 - \int_{-1}^1 \frac{d\mu}{2}\mu\tilde\Theta + \int_{-1}^1 \frac{d\mu}{2}\mu^2\,\tilde v_b \\ &= 0 - \int_{-1}^1 \frac{d\mu}{2}\mu\tilde\Theta + \frac{\tilde v_b}{3}. \end{aligned} \tag{2.186}$$

We now define the 1. moment of the temperature perturbation as

$$\tilde\Theta_1 \equiv i\int_{-1}^1 \frac{d\mu}{2}\mu\tilde\Theta, \tag{2.187}$$

which is a clear generalization of the zeroth moment $\tilde\Theta_0$ defined previously. In chapter 4 we will generalize to higher moments, but for now we use this definition to obtain

$$\int_{-1}^1 \frac{d\mu}{2}\mu\big[\tilde\Theta_0 - \tilde\Theta + \mu\,\tilde v_b\big] = i\tilde\Theta_1 + \frac{\tilde v_b}{3}. \tag{2.188}$$

Hence we finally arrive at

$$\begin{aligned}-\frac{a}{\rho_b} < C_{e\gamma}\,\mu p >_{pp'q'q} &= \frac{a\,n_e\sigma_T}{\rho_b} \times 4\rho_\gamma \times \left[i\tilde\Theta_1 + \frac{\tilde v_b}{3}\right] \\ &= \frac{a\,n_e\sigma_T 4\rho_\gamma}{3\rho_b}\left[3i\tilde\Theta_1 + \tilde v_b\right]. \end{aligned} \tag{2.189}$$

2.2.9 Baryon Velocity Equation

Combining equation (2.189) with equation (2.176) leads to

$$\dot{\tilde v}_b + \mathcal{H}\tilde v_b + i\,k\tilde\Psi = \frac{a\,n_e\sigma_T 4\rho_\gamma}{3\rho_b}\left[3i\tilde\Theta_1 + \tilde v_b\right]. \tag{2.190}$$

If we now introduce the definition

$$\frac{1}{R} \equiv \frac{4\rho_\gamma}{3\rho_b}. \tag{2.191}$$

The above equation becomes

$$\dot{\tilde{v}}_b + \mathcal{H}\,\tilde{v}_b = -i\,k\tilde{\Psi} + \frac{\dot{\tau}}{R}\left[3i\tilde{\Theta}_1 + \tilde{v}_b\right], \tag{2.192}$$

were I have used that $\dot{\tau} = a\,n_e\sigma_T$. This is the governing equation of the evolution of the baryon density. We will need this equation in the calculation of the acoustic oscillations in the cosmic plasma in chapter 4.
This concludes the calculations of the Boltzmann equations for baryons and photons. We have acquired almost all we need for an accurate description of the perturbations in the cosmic plasma. However we still have not considered the evolution of the potentials $\tilde{\Psi}$ and $\tilde{\Phi}$.

2.3 The Perturbed Einstein Equations

We end this chapter by addressing one aspect of cosmological perturbation theory, namely the Einstein equations of the perturbed metric $\delta G^{\nu}_{\mu} = 8\pi G\delta T^{\nu}_{\mu}$. These govern the evolution of the Bardeen potentials $\tilde{\Psi}$ and $\tilde{\Phi}$. The derivation is in principle straight forward but tedious, and unfortunately the lack of time does not permit a full calculation here. See [25, 10, 2] for detailed calculations. We will use the results from [10] in our work here. Essentially the equations reduce to

$$k^2\tilde{\Phi} + 3\mathcal{H}\left(\dot{\tilde{\Phi}} - \tilde{\Psi}\mathcal{H}\right) = 4\pi G a^2\left(\rho_{cdm}\delta + \rho_b\delta_b + 4\rho_\gamma\tilde{\Theta}_0\right) \tag{2.193}$$

$$\dot{\tilde{\Phi}} - a^2\mathcal{H} = \frac{4\pi G a^2}{ik}\left(\rho_{cdm}v + \rho_b v_b - 4i\rho_\gamma\tilde{\Theta}_1\right) \tag{2.194}$$

$$k^2(\tilde{\Phi} + \tilde{\Psi}) = -32\pi G a^2\rho_\gamma\tilde{\Theta}_2. \tag{2.195}$$

These correspond to the time-time, time-space and space-space components of the Einstein equations respectively and $\tilde{\Theta}_2$ is the 2. moment of the brightness function (see chapter 4 for details). Some comments are in order at this time. First we have made the simplification of ignoring the neutrinos. Their contribution is small in the time about recombination, so the error we are making is small. Secondly the right hand side of equation (2.195) constitutes what is called anisotropic stress, which in our case to the photon quadropole[3]. Hence when setting anisotropic stress to 0 we are ignoring the effects of $\tilde{\Theta}_2$, thus obtaining the identification $\tilde{\Psi} \simeq -\tilde{\Phi}$ we have used earlier.

[3] The neutrino quadropole will also contribute here in the massless neutrino setting.

Chapter 3

Inflation

3.1 Introduction

In this chapter we will give a short introduction to the Inflationary paradigm which was first introduced by Guth (1980) and has evolved considerably in the last 20 years[39]. Inflation has become a well known part of the standard cosmological model, where the quantum fluctuations at very early times set the seeds for structure formation. I will here present the simplest model of inflation, namely the single field description in which an analytic solution of the primordial perturbations can be found. In addition we will not cover all the the problems of the standard big bang theory that inflation solves, but will mainly focus on the *the Horizon problem*.

3.2 A few words on Cosmological Scales

Let us start by examining some aspects of the relevant scales in Cosmology. We will consider a flat universe. In a time increment dt, light travels a comoving distance dx given by

$$dx = \frac{dt}{a}. \tag{3.1}$$

We can integrate this equation to obtain the total comoving distance light could have traveled

$$x = \int_0^t \frac{dt'}{a'}, \tag{3.2}$$

The quantity x defines the *comoving horizon*. Regions that are separated a distance larger than x could therefore not have contact. Observe that this expression is precisely how we defined conformal time η. Hence we have a physical interpretation of conformal time simply being the comoving horizon. We can rewrite the above integral by recalling that

$$\begin{aligned}\frac{da}{dt} &= aH \\ dt &= \frac{1}{aH}da. \end{aligned} \tag{3.3}$$

This gives

$$\eta = \int_0^a \frac{da'}{a'}\frac{1}{a'H'}. \tag{3.4}$$

The fraction $(aH)^{-1}$ is the *comoving Hubble radius*. This distance is a measure of the connection between particles. The difference between the comoving Hubble radius and the comoving horizon η is subtle. Particles separated by distances larger than $(aH)^{-1}$ can not communicate *now*, while particles separated by η could never have communicated. If $\eta \gg (aH)^{-1}$ now, it is still possible that that the particles could communicate at one time in the past, if the comoving Hubble radius was much larger than it's present value.

3.3 The Horizon Problem

Let us look now make an order-of-magnitude calculation of the casually connected regions of the sky. First the comoving horizon at recombination is simply η_*. At the present time the comoving size of the regions we observe the CMB is $\eta_0 - \eta_*$. For simplicity let us assume that the universe is matter dominated the whole time from decoupling to the present. Then we have from [32]

$$\eta_* \simeq 3t_0^{\frac{2}{3}}t_*^{\frac{1}{3}} \tag{3.5}$$

$$\eta_0 - \eta_* \simeq 3t_0\left[1 - \left(\frac{t_*}{t_0}\right)^{\frac{1}{3}}\right] \simeq 3t_0. \tag{3.6}$$

The ratio of these two quantities is

$$\frac{\eta_*}{\eta_0 - \eta_*} \simeq \frac{3t_0^{\frac{2}{3}}t_*^{\frac{1}{3}}}{3t_0} = \left(\frac{t_*}{t_0}\right)^{\frac{1}{3}} \simeq \left(\frac{10^5}{10^{10}}\right)^{\frac{1}{3}} \simeq 0.01 \tag{3.7}$$

This means that the casually connected regions of the sky subtend an angle of approximately $1°$ [39]. This is of course inconsistent with the observed CMB, being isotropic on the whole sky. It is this that we call the *Horizon problem*.

3.4 Solution of the Horizon Problem

How does one come to a solution of the Horizon problem. One possibility is that the comoving Hubble radius $(aH)^{-1}$ decreased in an early period of time. Connected regions at one time would then cease to be in contact as $(aH)^{-1}$ decreases. We will refer to the period as *the epoch of Inflation*. Hence when inflation ends, and as the universe becomes radiation dominated, the comoving Hubble radius starts to increase again. The connected regions before inflation will then gradually reenter the horizon as the universe expands, thus explaining, in at this stage, a qualitative way the large scale isotropy of the CMB. Let us now see what implications this simple assumption will lead to.

3.4.1 The Accelerating Universe

We have defined the inflationary epoch as the era when $(aH)^{-1}$ is decreasing (if possible). Let us see what this will imply. The assumption is equivalent to

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{aH}\right) &< 0 \\ \frac{d}{dt}\left(\frac{1}{da/dt}\right) &< 0 \\ -\frac{d^2a/dt^2}{(da/dt)^2} &< 0 \\ \frac{d^2a}{dt^2} &> 0. \end{aligned} \tag{3.8}$$

Thus the universe must be in an accelerated state of expansion during inflation.

3.4.2 Negative Pressure

The question now is what sort of substance can cause an accelerated expansion. To answer this we must return to the unperturbed zero order Einstein equations. These are

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho \tag{3.9}$$

$$\frac{1}{a^2}\frac{d^2a}{dt^2} + \frac{1}{2}\left(\frac{1}{a}\frac{da}{dt}\right)^2 = -4\pi GP, \tag{3.10}$$

where P and ρ are the zero order pressure and density respectively. Combining these two equations gives us

$$\frac{1}{a^2}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3P)\,. \tag{3.11}$$

Demanding an accelerated state leads to

$$\rho + 3P < 0, \tag{3.12}$$

or simply $P < -\rho/3$. Since energy density is by assumption always positive, we are forced to conclude that the species responsible for inflation has a negative pressure. Recall that baryonic matter has $P = 0$ and radiation has $P = \rho/3$. Hence this substance must be completely different from what we have already discovered. Before we move on to model this substance, let us see how we can quantify the amount of inflation required for solving the horizon problem.

3.4.3 The Amount of Inflation: e-foldings

Let us now see how inflation can account for the large scale homogeneity of the CMB. To do this we will make an order of magnitude estimate comparing the comoving Hubble radius at the end of inflation and at the present time. The ratio of these quantities is

$$\frac{(H_e a_e)^{-1}}{(H_0 a_0)^{-1}} = \frac{H_0 a_0}{H_e a_e}. \tag{3.13}$$

Here we have defined $a_e \equiv a(t_e)$ where t_e is the time when inflation ends. For simplicity we will assume that the universe is completely radiation dominated after inflation until the present time. In a radiation dominated universe $\rho \sim a^{-4}$. Using equation (3.9) we get

$$H \sim \sqrt{\rho} \sim a^{-2}. \tag{3.14}$$

Hence $H_e \sim a_e^{-2} H_0$. Using this result (with $a_0 = 1$) gives

$$\frac{H_0 a_0}{H_e a_e} \sim \frac{a_e^2 H_e}{H_e a_e} \sim a_e. \tag{3.15}$$

The typical energy scale[27] at inflation is $T \sim 10^{15} GeV$. With $T_0 \sim 10^{-4} eV$, we obtain

$$a_e \sim \frac{T_0}{10^{15}\text{GeV}} \sim \frac{10^{-4} eV}{10^{15}\text{GeV}} \sim 10^{-28}. \tag{3.16}$$

Hence for inflation to work the comoving Hubble radius must decrease by 28 orders of magnitude. We are dealing with large numbers here, it is therefore convenient to define the number of *e-foldings* $N(t)$ as the logarithm of the inverse of the above ratio.

$$N(t) \equiv \ln \frac{H_e a_e}{Ha}. \tag{3.17}$$

This is a measure of the amount of inflation. The Hubble factor during inflation varies much slower than the expansion factor a, hence we can take H to be constant whenever it multiplies a. We will in fact use this approximation a great deal in the calculations in the coming sections. The nearly constancy of H has made it more common to define the number of e-foldings as

$$N(t) \equiv \ln \frac{a_e}{a} = \int_t^{t_e} H(t')dt'. \tag{3.18}$$

With the estimation we have recently carried out, we would need $N \sim \ln 10^{28} \sim 60$ e-foldings to solve the horizon problem[10].

3.5 The Inflaton Field

In this section we will implement a scalar field $\phi(\vec{x}, t)$ to try to describe the substance responsible of inflation. We will use a field theoretical approach, characterized by a Lagrangian density $\mathcal{L}$ for ϕ, which is often referred to as the *inflaton field*. There are many different ways to describe the inflaton field, we will opt for the easiest approach here, namely a single field description. The Lagrangian of the single scalar field [27] is

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} - V(\phi). \tag{3.19}$$

where $V(\phi)$ is the potential of the field ϕ. We will at first be regarding a field which is mostly homogenous in space leaving the study of first order perturbation of the field to section 3.7.2 .

3.5.1 Density and Pressure of the Inflaton Field

As an introduction to the study of the field, let us determine its pressure and density. We will then need the energy- momentum tensor of a classical field [27].The energy-momentum tensor is given by

$$T_{\mu\nu} = -2\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} + g_{\mu\nu}\mathcal{L}. \tag{3.20}$$

Inserting the expression for $\mathcal{L}$ leads to

$$T_{\mu\nu} = \frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta} + V(\phi)\right]. \tag{3.21}$$

Recall that the simplest form of the energy-momentum tensor is the one with mixed indices whose components is $T^\mu{}_\nu = diag(-\rho, P, P, P)$. Raising an index in the above equation leads us to

$$T^\mu{}_\nu = g^{\mu\alpha}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\nu} - g^\mu{}_\nu\left[\frac{1}{2}g^{\alpha\beta}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta} + V(\phi)\right]. \tag{3.22}$$

We want to find an expression for the pressure P and the density ρ in terms of the field variables. We are at the present studying the zero order homogenous part of the field where the spatial derivatives vanish, and we will be using the unperturbed metric $g_{\mu\nu} = diag(-1, a^2, a^2, a^2)$. Since $T^0{}_0 = -\rho$, we get

$$\begin{aligned}
T^0{}_0 &= g^{0\alpha}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^0} - g^0{}_0\left[\frac{1}{2}g^{\alpha\beta}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta} + V(\phi)\right] \\
&= g^{00}\frac{\partial\phi}{\partial x^0}\frac{\partial\phi}{\partial x^0} - \left[\frac{1}{2}g^{00}\frac{\partial\phi}{\partial x^0}\frac{\partial\phi}{\partial x^0} + V(\phi)\right] \\
&= -\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - V(\phi) \\
&= -\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - V(\phi).
\end{aligned} \tag{3.23}$$

Hence the field density becomes

$$\rho = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + V(\phi). \tag{3.24}$$

We find the pressure in a similar way, calculating T^i_i which is equal to P. Choosing $T^1{}_1$ we obtain

$$\begin{aligned}
T^1{}_1 &= g^{10}\frac{\partial\phi}{\partial x^0}\frac{\partial\phi}{\partial x^1} - g^1{}_1\left[\frac{1}{2}g^{\alpha\beta}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta} + V(\phi)\right] \\
&= 0 - \left[-\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + V(\phi)\right] \\
&= \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - V(\phi).
\end{aligned} \tag{3.25}$$

Hence the pressure of the field is

$$P = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - V(\phi). \tag{3.26}$$

We will return to the study of these in section 3.6.

3.5.2 Field Equations

We will here find an expression for the field equations of the homogenous inflaton field. We will use the Euler-Lagrange equation [10] given by

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{3.27}$$

We will however make no assumption of spatial independence in the below calculations as the resulting equation will be useful when finding the first order perturbations of ϕ. It will in addition be most useful to obtain the field equation in conformal time, hence we will take the metric to be $g_{\mu\nu} = a^2\mathrm{diag}(-1, 1, 1, 1)$. Using equation (3.19) we obtain

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -g^{\mu\nu}\partial_\nu\phi \tag{3.28}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -V'(\phi). \tag{3.29}$$

This leads to

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) &= -\partial_\mu g^{\mu\nu}\partial_\nu\phi - g^{\mu\nu}\partial_\mu\partial_\nu\phi \\ &= -\partial_0 g^{00}\,\partial_0\phi + \frac{1}{a^2}\ddot{\phi} - \frac{1}{a^2}\nabla^2\phi \\ &= 2\frac{\dot{a}}{a^3}\dot{\phi} + \frac{1}{a^2}\ddot{\phi} - \frac{1}{a^2}\nabla^2\phi. \end{aligned} \tag{3.30}$$

The Euler-Lagrange equation becomes

$$\begin{aligned} 2\frac{\dot{a}}{a^3}\dot{\phi} + \frac{1}{a^2}\ddot{\phi} - \frac{1}{a^2}\nabla^2\phi + V'(\phi) &= 0 \\ \ddot{\phi} + 2\frac{\dot{a}}{a}\dot{\phi} + a^2V'(\phi) - \nabla^2\phi &= 0. \end{aligned} \tag{3.31}$$

Using that $\dot{a} = a^2H$, the homogenous equation of the inflaton field becomes

$$\ddot{\phi} + 2aH\dot{\phi} + a^2V'(\phi) = 0. \tag{3.32}$$

This is the equation describing the dynamics of ϕ. Given a potential V, we could in principle solve this equation, although not many analytic solutions are known. We will in the next section study the *slow roll approximation* of inflation which has become an integral part of inflationary theory. For this we will need the above equation with respect to ordinary time. This is simply[27]

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + V'(\phi) = 0. \tag{3.33}$$

3.6 Slow Roll Inflation

Let us revisit the negative pressure criteria we found in section 3.4.2. This condition gives

$$P = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - V(\phi) < 0. \tag{3.34}$$

This corresponds to a field configuration with more potential than kinetic energy[39]. In a sense the field is trapped in a local potential minimum, also known as a *false vacuum*, where the global minimum is the true vacuum. Sooner or later the field must reach its' true vacuum state, but if the field is trapped, it must change very slowly with time. This is the basis of the *slow roll approximation*, where we demand that the field "rolls slowly" from its' false vacuum state to the true global minimum of the potential. Moving too fast may violate the negative pressure condition. There is also the notion that the field quantum mechanically tunnels to this state, but I will not cover this here. Since ϕ varies very slowly, equation (3.9) becomes

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho \\ &\approx \frac{8\pi G}{3} V(\phi) \end{aligned} \tag{3.35}$$

This also shows that H varies very slowly during inflation. We can also omit the $\ddot{\phi}$ term in equation (3.33). This gives

$$3H\frac{d\phi}{dt} \approx -V'(\phi). \tag{3.36}$$

3.6.1 Slow Roll Parameters

It has become customary and useful to define two parameters to quantify slow-roll inflation[27]. The first of which is

$$\epsilon \equiv \frac{d}{dt}\left(\frac{1}{H}\right) = -\frac{1}{H^2}\frac{dH}{dt} = -\frac{\dot{H}}{aH^2}. \tag{3.37}$$

This quantity is related to our work in the following way. Recall that during inflation we require $d^2a/dt^2 > 0$, thus

$$\frac{1}{a}\frac{d^2a}{dt^2} = \frac{dH}{dt} + H^2 > 0. \tag{3.38}$$

This gives

$$-\frac{1}{H^2}\frac{dH}{dt} < 1. \tag{3.39}$$

Hence during inflation, $\epsilon < 1$. Many inflationary models use $\epsilon = 1$ as an inflation ending criterium. We also see that ϵ is always positive during inflation since dH/dt is negative. For a slowly rolling field we require $\epsilon \ll 1$.
The second parameter we will use is defined as

$$\delta \equiv \frac{1}{8\pi G} \frac{V''}{V}. \tag{3.40}$$

Observe that this parameter is more commonly referred to as η in the literature, which we cannot use since this is our conformal time symbol. We require $|\delta| \ll 1$. One may also express ϵ in terms of the field potential $V(\phi)$ in the following manner. Firstly if we differentiate equation (3.35), we get

$$6H\frac{dH}{dt} = 8\pi G V'(\phi)\frac{d\phi}{dt}. \tag{3.41}$$

Using equation (3.36) to solve for $d\phi/dt$ we get

$$\frac{d\phi}{dt} \approx \frac{-V'(\phi)}{3H}. \tag{3.42}$$

This leads to

$$18H\frac{dH}{dt} = -8\pi G\frac{(V'(\phi))^2}{H}. \tag{3.43}$$

Dividing by the square of equation (3.35) we get

$$\frac{18H^2\frac{dH}{dt}}{9H^4} = -8\pi G\frac{(V')^2}{64\pi^2 G^2 V^2}. \tag{3.44}$$

This leads to

$$-\frac{\frac{dH}{dt}}{H^2} = \frac{(V')^2}{16\pi G V^2}. \tag{3.45}$$

Hence we obtain

$$\epsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2. \tag{3.46}$$

We will in the upcoming calculations express encountered quantities by the slow roll parameters. It turns out that this will simplify our work considerably.

3.7 Cosmological Density Perturbations During Inflation

We will in this section find the equation for the perturbations in the inflaton field. This we will accomplish by splitting up the field into a zero order homogenous part and a first order perturbative part $\delta\phi$, and use the field equation we found in section 3.5.2 to find the equation for the perturbations. The perturbations in $\delta\phi$ are thought to be partially quantum mechanical in nature. We will thus define the *power spectrum* of these fluctuations and see that they are directly related to the metric perturbations, which are believed to set the seeds of the large scale structure in the universe.

3.7.1 Rescaling Conformal Time

In this framework, anisotropic stress is completely negligible, hence $\Psi = -\Phi$. Recall also that $k = 2\pi/\lambda$, where λ is the comoving wavelength of a given k-mode. If the ratio $k\eta$ is much less than 1, the mode in question has a wavelength larger than the horizon[10]. Equivalently $k\eta = 1$ corresponds to horizon sized modes and $k\eta > 1$ translates into sub-horizon modes. Observe also that of all cosmological epochs, η undergoes the greatest increase during inflation. It has thus been customary to redefine η by subtracting off the primordial part η_{prim}. This implies that

$$\eta \equiv \int_{t_e}^{t} \frac{1}{a(t')} dt'. \tag{3.47}$$

where t_e is the time when inflation ends. The comoving horizon has now become $\eta_{prim} + \eta$. One can also see that η is negative during inflation. Since H varies very slowly during inflation, we can get a rough estimate of η. We will later refine this estimate to include the slow roll parameters.

$$\begin{aligned}
\eta &= \int_{t_e}^{t} \frac{1}{a(t')} dt' \\
&= \int_{a_e}^{a} \frac{1}{Ha'^2} da' \\
&\approx \frac{1}{H} \int_{a_e}^{a} \frac{1}{a'^2} da' \\
&= -\frac{1}{H} \left[\frac{1}{a'}\right]_{a_e}^{a} \\
&= -\frac{1}{H} \left[\frac{1}{a} - \frac{1}{a_e}\right].
\end{aligned} \tag{3.48}$$

Since $a_e \gg a$ we get

$$\eta \approx -\frac{1}{aH}. \tag{3.49}$$

3.7.2 Perturbative Inflaton Equation

We will now find the equation of the first order perturbation of the inflaton field $\phi \to \phi + \delta\phi$. The complete field equation for ϕ is

$$\ddot{\phi} + 2aH\dot{\phi} + a^2 V'(\phi) - \nabla^2\phi = 0\,. \tag{3.50}$$

Observe that this procedure will give a slight error in the resulting equation since we are in principle neglecting the effect of the metric perturbation Ψ on $\delta\phi$. We could account for this by going to the perturbative Einstein equation $\delta G^{\mu}{}_{\nu} = 4\pi G \delta T^{\mu}{}_{\nu}$, which would give us a set of coupled equations for Ψ and $\delta\phi$. It turns out that this rather tedious approach will give us roughly the same equation for $\delta\phi$, give or take a factor proportional to $\epsilon\,\delta\phi$. This will in turn infer a slight modification in the spectral index n to be defined later. Hence the error we are making is quite small with our approach. Thus let us set $\phi \to \phi + \delta\phi$ in equation (3.50). We get

$$\begin{aligned} \ddot{\phi} + \ddot{\delta\phi} + 2aH(\dot{\phi} + \dot{\delta\phi}) + a^2 V'(\phi + \delta\phi) - \nabla^2(\phi + \delta\phi) = 0 \\ \ddot{\phi} + 2aH\dot{\phi} + a^2 V'(\phi + \delta\phi) + \ddot{\delta\phi} + 2aH\dot{\delta\phi} - \nabla^2\delta\phi = 0. \end{aligned} \tag{3.51}$$

The derivative of ϕ can be expanded to first order as

$$V'(\phi + \delta\phi) \approx V'(\phi) + \delta\phi V''(\phi). \tag{3.52}$$

This gives

$$\ddot{\phi} + 2aH\dot{\phi} + a^2 V'(\phi) + a^2 V''(\phi)\delta\phi + \ddot{\delta\phi} + 2aH\dot{\delta\phi} - \nabla^2\delta\phi = 0\,. \tag{3.53}$$

The first three terms add up to zero by virtue of the zero order equation. We then obtain

$$\ddot{\delta\phi} + 2aH\dot{\delta\phi} - \nabla^2\delta\phi + a^2 V''(\phi)\,\delta\phi = 0\,. \tag{3.54}$$

Going to Fourier space, we finally get

$$\ddot{\delta\phi} + 2aH\dot{\delta\phi} + \left(k^2 + a^2 V''\right)\delta\phi = 0\,. \tag{3.55}$$

This is the equation of the perturbations induced by the field.

3.7.3 Solving the Perturbation Equation

We will now solve the perturbation equation obtained in the previous section. First, we redefine the field[39] by $h \equiv a\delta\phi$. Differentiation gives

$$\delta\phi = \frac{h}{a} \tag{3.56}$$

$$\delta\dot{\phi} = \frac{\dot{h}}{a} - \frac{\dot{a}}{a^2}h \tag{3.57}$$

$$\delta\ddot{\phi} = \frac{\ddot{h}}{a} - 2\frac{\dot{a}}{a^2}\dot{h} - \frac{\ddot{a}}{a^2}h + 2\frac{\dot{a}^2}{a^3}h\,. \tag{3.58}$$

Inserting these into equation (3.55) leads to

$$\frac{\ddot{h}}{a} - 2\frac{\dot{a}}{a^2}\dot{h} - \frac{\ddot{a}}{a^2}h + 2\frac{\dot{a}^2}{a^3}h + 2\frac{\dot{a}}{a}\left(\frac{\dot{h}}{a} - \frac{\dot{a}}{a^2}h\right) + \left(k^2 + a^2V''\right)\frac{h}{a} = 0\,. \tag{3.59}$$

This gives

$$\frac{\ddot{h}}{a} + \left(k^2 + a^2V'' - \frac{\ddot{a}}{a}\right)\frac{h}{a} = 0\,. \tag{3.60}$$

Multiplication by a finally leads to

$$\ddot{h} + \left(k^2 + a^2V'' - \frac{\ddot{a}}{a}\right)h = 0\,. \tag{3.61}$$

We can simplify this equation further by introducing the slow roll parameters. First we can note that

$$\begin{aligned}\frac{d}{d\eta}\left(\frac{1}{aH}\right) &= -\frac{1}{a^2H^2}\frac{d}{d\eta}(aH)\\ &= -\frac{1}{a^2H^2}\left(\dot{a}H + a\dot{H}\right)\\ &= -\frac{\dot{a}}{a^2H} - \frac{\dot{H}}{aH^2}\,.\end{aligned} \tag{3.62}$$

The last term is simply ϵ. Since $\dot{a} = a^2H$ we get

$$\frac{d}{d\eta}\left(\frac{1}{aH}\right) = \epsilon - 1\,. \tag{3.63}$$

We can use this equation to approximate the $\ddot{a}/a$ term. Since $\frac{1}{aH} = \frac{a}{\dot{a}}$, we obtain

$$\begin{aligned}
\frac{d}{d\eta}\left(\frac{a}{\dot{a}}\right) &= \epsilon - 1 \\
1 - \frac{a\ddot{a}}{\dot{a}^2} &= \epsilon - 1 \\
2 - \epsilon &= \frac{a^2}{\dot{a}^2}\ddot{a} \\
2 - \epsilon &= \frac{1}{a^2H^2}\,\frac{\ddot{a}}{a}\,. \qquad (3.64)
\end{aligned}$$

Thus we get

$$\frac{\ddot{a}}{a} = a^2H^2(2-\epsilon)\,. \qquad (3.65)$$

We could now have approximated $a^2H^2 \sim \eta^{-2}$ as suggested in section 3.7.1, but refining this approximation will give us a less erroneous solution. Assuming a slowly varying ϵ, we can integrate equation (3.63) to obtain

$$\begin{aligned}
\frac{1}{aH} &= \eta(\epsilon - 1) \\
\frac{1}{a^2H^2} &= \eta^2(\epsilon - 1)^2 \\
\frac{1}{a^2H^2} &\approx \eta^2(1 - 2\epsilon)\,. \qquad (3.66)
\end{aligned}$$

which gives

$$a^2H^2 = \frac{1}{\eta^2(1-2\epsilon)} \approx \frac{1}{\eta^2}(1+2\epsilon)\,. \qquad (3.67)$$

Combining this with equation (3.65) leads to

$$\frac{\ddot{a}}{a} \approx \frac{1}{\eta^2}(1+2\epsilon)(2-\epsilon) \approx \frac{1}{\eta^2}(2+3\epsilon)\,. \qquad (3.68)$$

Working now with the a^2V'' term, we have to first order

$$a^2V'' = a^2V\frac{V''}{V} \approx a^2\frac{3H^2}{8\pi G}\frac{V''}{V} = 3a^2H^2\delta \approx \frac{3}{\eta^2}(1+2\epsilon)\delta \approx \frac{3\delta}{\eta^2}\,. \qquad (3.69)$$

where I have used $8\pi G \approx 3H^2$ (valid only in the slow roll limit). Inserting the above results into equation (3.61) gives

$$\begin{aligned}
\ddot{h} + \left(k^2 + \frac{3\delta}{\eta^2} - \frac{1}{\eta^2}(2+3\epsilon)\right)h &= 0 \\
\ddot{h} + \left(k^2 - \frac{1}{\eta^2}\left(2 - 3\delta + 3\epsilon\right)\right)h &= 0\,. \qquad (3.70)
\end{aligned}$$

Defining $\nu^2 = \frac{9}{4} - 3\delta + 3\epsilon$ we get

$$\ddot{h} + \left(k^2 + \frac{1}{\eta^2}\left(\frac{1}{4} - \nu^2\right)\right) h = 0\,. \tag{3.71}$$

Assuming now a slow variation of ν, the general solution of this equation can be expressed by the *Hankel functions* $H_\nu^{(1)}(-k\eta)$ and $H_\nu^{(2)}(-k\eta)$ of the 1. and 2. kind[6], see section C.1 for details. Hence our general solution is

$$h = \sqrt{-\eta}\left[c_1 H_\nu^{(1)}(-k\eta) + c_2 H_\nu^{(2)}(-k\eta)\right]\,, \tag{3.72}$$

where c_1 and c_2 are integration constants, To find these we can first note that $-k\eta \gg 1$, in sub-horizon regions, equation (3.71) reduces to the well known quantum mechanical oscillator equation[39]. The solution in this limit must thus take on the form $\frac{e^{-ik\eta}}{\sqrt{2k}}$. Studying the large valued limit of the Hankel functions, we see that this automatically excludes $H_\nu^{(2)}(-k\eta)$ which is of the order $\sim e^{ik\eta}$. Hence $c_2 = 0$. The large valued limit of $H_\nu^{(1)}(-k\eta)$ is

$$H_\nu^{(1)}(-k\eta) \simeq \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{-k\eta}} e^{-ik\eta} e^{-i(\nu+\frac{1}{2})\frac{\pi}{2}}\,. \tag{3.73}$$

For h we now have

$$h = \sqrt{-\eta}\, c_1 H_\nu^{(1)}(-k\eta) \stackrel{!}{=} \frac{e^{-ik\eta}}{\sqrt{2k}}\,. \tag{3.74}$$

Combining equation (3.73) with the above condition gives

$$\sqrt{-\eta}\, c_1 \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{-k\eta}} e^{-ik\eta} e^{-i(\nu+\frac{1}{2})\frac{\pi}{2}} = \frac{e^{-ik\eta}}{\sqrt{2k}}\,. \tag{3.75}$$

Solving for c_1 leads to

$$c_1 = \frac{\sqrt{\pi}}{2} e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}\,. \tag{3.76}$$

Thus our solution becomes

$$h = \frac{\sqrt{\pi}}{2}\sqrt{-\eta}\, e^{i(\nu+\frac{1}{2})\frac{\pi}{2}} H_\nu^{(1)}(-k\eta)\,. \tag{3.77}$$

To go further, let us study the super-horizon limit of the above expression, i.e. when $-k\eta \ll 1$. The Hankel function at this limit reduces to

$$H_\nu^{(1)}(-k\eta) \simeq \frac{2^\nu}{i}\,\Gamma(\nu)\,(-k\eta)^{-\nu} \tag{3.78}$$

This leads us into

$$
\begin{aligned}
h &\simeq \frac{\sqrt{\pi}}{2}\sqrt{-\eta}\, e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}\frac{2^{\nu}}{i}\,\Gamma(\nu)\,(-k\eta)^{-\nu} \\
&= \frac{2^{\nu-\frac{1}{2}}}{\sqrt{2\pi k^3}} e^{i(\nu-\frac{1}{2})\frac{\pi}{2}} k^{\frac{3}{2}-\nu}(-\eta)^{\frac{1}{2}-\nu}\Gamma(\nu)\,. \qquad (3.79)
\end{aligned}
$$

Recall that to a first approximation, $-\eta \sim (aH)^{-1}$, and $\Gamma(\nu) \simeq \Gamma(3/2) = \sqrt{\pi}/2$ since ϵ and δ are quite small. This gives

$$
\begin{aligned}
h &\simeq \frac{2^{\nu-\frac{1}{2}}}{\sqrt{2\pi k^3}} e^{i(\nu-\frac{1}{2})\frac{\pi}{2}} k^{\frac{3}{2}-\nu}\left(\frac{1}{aH}\right)^{\frac{1}{2}-\nu}\frac{\sqrt{\pi}}{2} \\
&= \frac{2^{\nu-\frac{3}{2}}}{\sqrt{2k^3}} e^{i(\nu-\frac{1}{2})\frac{\pi}{2}} k^{\frac{3}{2}-\nu} aH \left(\frac{1}{aH}\right)^{\frac{3}{2}-\nu} \\
&= \frac{2^{\nu-\frac{3}{2}}}{\sqrt{2k^3}} e^{i(\nu-\frac{1}{2})\frac{\pi}{2}} aH \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \\
&\simeq \frac{e^{i(\nu-\frac{1}{2})\frac{\pi}{2}}}{\sqrt{2k^3}} aH \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}\,, \qquad (3.80)
\end{aligned}
$$

where in the last line I have approximated $2^{\nu-\frac{3}{2}} \simeq 1$. Thus the inflaton perturbation $\delta\phi$ is given by

$$
\delta\phi = \frac{H e^{i(\nu-\frac{1}{2})\frac{\pi}{2}}}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}\,. \qquad (3.81)
$$

We will in the coming section only be interested in the amplitude $|\delta\phi|$ of the inflaton perturbation. This is

$$
|\delta\phi| = \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}\,. \qquad (3.82)
$$

We will now see how these fluctuations relate to the curvature perturbations which are responsible for the large scale structure in the universe[39].

3.8 Primordial Metric Perturbations

We will now define the *power spectrum* of the metric perturbation $\Phi = -\Psi$. It is defined[10]as

$$
\begin{aligned}
<\hat{\Psi}^{\dagger}\hat{\Psi}> &= (2\pi)^3 P_{\Psi}\delta^3(\vec{k}-\vec{k}') \\
&= (2\pi)^3|\Psi|^2\delta^3(\vec{k}-\vec{k}')\,. \qquad (3.83)
\end{aligned}
$$

Here $\hat{\Psi}$ is viewed as a quantum mechanical operator. The appearance of the delta function is a consequence of the independent evolution of different k-modes. We can in a similar fashion define the power spectrum $P_{\delta\phi}$ for the inflaton perturbation

$$\begin{aligned} < \hat{\delta\phi}^{\dagger} \hat{\delta\phi} > &= (2\pi)^3 P_{\delta\phi}\, \delta^3(\vec{k} - \vec{k}') \\ &= (2\pi)^3 |\delta\phi|^2 \delta^3(\vec{k} - \vec{k}')\,. \end{aligned} \tag{3.84}$$

Observe that some authors include an extra factor of k^3 in the definition of the power spectrum like in [16]. We will now undertake the task of relating $P_\Psi = P_\Phi$ with $P_{\delta\phi}$. Later in this chapter we shall relate P_Ψ with the *matter power spectrum* $\mathcal{P}(k)$ defined as

$$< \hat{\delta}^{\dagger}(\vec{k})\hat{\delta}(\vec{k}') > = (2\pi)^3 \mathcal{P}(k) \delta^3(\vec{k} - \vec{k}'). \tag{3.85}$$

where δ is the matter density contrast of the total matter component of the universe defined earlier in chapter 2.

3.8.1 The Curvature Perturbation ζ

A straightforward way of relating Ψ to $\delta\phi$ that might seem strange at first is to define a function ζ by

$$\zeta = -\frac{ik_i \delta T^0{}_i H}{k^2(\rho + P)} - \Psi, \tag{3.86}$$

where δT^0_i is the ${}^0{}_i$-component of the perturbed energy-momentum tensor. It turns out that ζ is in fact a gauge invariant quantity, first identified by Bardeen [4] which is a constant in time. We will take this as given at this time, leaving the proof of this to section 3.8.4. Let us see what this fact implies.
For modes at horizon crossing ($aH = k$), Ψ is completely negligible[39]. In addition we have that

$$P + \rho = \frac{\dot{\phi}^2}{a^2}\,. \tag{3.87}$$

To find δT^0_i we can use equation (3.22)

$$\begin{aligned} T^0{}_i &= g^{00} \frac{\partial \phi}{\partial x^0} \frac{\partial \phi}{\partial x^i} - g^0{}_i \left[\frac{1}{2} g^{\alpha\beta} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\beta} + V(\phi) \right] \\ &= -\frac{\dot{\phi}}{a^3} \frac{\partial \phi}{\partial x^i}\,. \end{aligned} \tag{3.88}$$

This is of course 0 for the homogenous part of the field. Letting $\phi \to \phi + \delta\phi$ leads to

$$\begin{aligned} T^0_{\ i} + \delta T^0_{\ i} &= -\frac{\dot{\phi} + \dot{\delta\phi}}{a^3}\frac{\partial\phi}{\partial x^i} - \frac{\dot{\phi} + \dot{\delta\phi}}{a^3}\frac{\partial\delta\phi}{\partial x^i} \\ &= T^0_{\ i} - \frac{\dot{\delta\phi}}{a^3}\frac{\partial\phi}{\partial x^i} - \frac{\dot{\phi}}{a^3}\frac{\partial\delta\phi}{\partial x^i} \, . \end{aligned} \tag{3.89}$$

Subtracting off the zero order part and retaining only the 1. order term, we obtain

$$\delta T^0_{\ i} = -\frac{\dot{\phi}}{a^3}\frac{\partial\delta\phi}{\partial x^i} \, . \tag{3.90}$$

In Fourier space this expression becomes

$$\delta T^0_{\ i} = -\frac{ik_i\dot{\phi}\,\delta\phi}{a^3} \, . \tag{3.91}$$

Using this result equation (3.86) reduces to

$$\begin{aligned} \zeta &= -\frac{ik_i H\left(-\frac{ik_i\dot{\phi}\,\delta\phi}{a^3}\right)}{k^2\frac{\dot{\phi}^2}{a^2}} \\ &= -\frac{ak^2\dot{\phi}\,\delta\phi H}{k^2\dot{\phi}^2} \\ &= -\frac{a\delta\phi H}{\dot{\phi}} \, . \end{aligned} \tag{3.92}$$

Observe that this expression should be evaluated at the time of horizon crossing. Let us now see what happens to ζ as inflation ends.

3.8.2 Perturbed Energy-Momentum Tensor for Radiation

As inflation ends, the universe becomes dominated by radiation. We will now evaluate ζ at this epoch, and for this we will need the expression for the energy-momentum tensor. In general it is given by[10]

$$T^\mu_{\ \nu} = g_i \int \frac{dP_1 dP_2 dP_3}{(2\pi)^3}\,(-|g|)^{-\frac{1}{2}}\,\frac{P^\mu P_\nu}{P_0} f_i \, , \tag{3.93}$$

where f_i is the distribution function for species i. For the photon, $g_i = 2$. We are interested in $T^0_{\ i}$, for which the above expression reduces to

$$T^0_{\ i} = 2a \int \frac{d^3p}{(2\pi)^3} p_i f \, , \tag{3.94}$$

where p_i is the photon momentum. We shall now perturb the above expression to obtain $\delta T^0_{\ i}$. In chapter 2 we saw that the first order perturbation of the distribution function is

$$f = f_o - p\frac{\partial f_0}{\partial p}\Theta. \tag{3.95}$$

This gives

$$T^0_{\ i(0)} + \delta T^0_{\ i} = 2a\int \frac{d^3p}{(2\pi)^3}p_i f_0 - 2a\int \frac{d^3p}{(2\pi)^3}pp_i\frac{\partial f_0}{\partial p}\Theta\,. \tag{3.96}$$

Subtracting off the zero order part we get

$$\delta T^0_{\ i} = -2a\int \frac{d^3p}{(2\pi)^3}p^2 n_i\frac{\partial f_0}{\partial p}\Theta\,. \tag{3.97}$$

Recalling that $k_i n_i = \mu k$, we can contract with ik_i to obtain

$$\begin{aligned} ik_i\delta T^0_{\ i} &= -2ak\int \frac{d^3p}{(2\pi)^3}ip^2\mu\frac{\partial f_0}{\partial p}\Theta \\ &= -2ak\int_0^\infty dpp^4\frac{\partial f_0}{\partial p}\frac{1}{-i}\int \frac{d\Omega}{(2\pi)^3}\mu\Theta \\ &= -2ak\int_0^\infty \frac{dp}{2\pi^2}p^4\frac{\partial f_0}{\partial p}\frac{1}{-i}\int_{-1}^1 \frac{d\mu}{2}\mu\Theta\,. \end{aligned} \tag{3.98}$$

We have calculated the momentum integral before, it is simply $-2\rho_r$. The angle integral is by definition the dipole Θ_1. We thus get

$$ik_i\delta T^0_{\ i} = 4ak\rho_r\Theta_1\,. \tag{3.99}$$

3.8.3 The Metric Power Spectrum

Obtaining a result for $ik_i\delta T^0_{\ i}$ at the end of inflation allows us to find an expression for ζ, which we can compare to the one we found in section 3.8.1. Using the fact that when radiation dominates, $P+\rho = 4\rho_r/3$, equation (3.86) becomes

$$\begin{aligned} \zeta &= -\frac{4ak\rho_r\Theta_1 H}{k^2\frac{4}{3}\rho_r} - \Psi \\ &= -\frac{3a\Theta_1 H}{k} - \Psi \end{aligned} \tag{3.100}$$

A result obtained from [10] is that for adiabatic initial conditions we have that $\Theta_1 = \frac{k\Psi}{6aH}$. This gives us

$$\zeta = -\frac{1}{2}\Psi - \Psi = -\frac{3}{2}\Psi\,. \tag{3.101}$$

Since ζ is constant in time, we can finally relate ψ to ζ at horizon crossing. This is

$$\Psi = -\frac{2}{3}\zeta = \frac{2}{3}\frac{aH\delta\phi}{\dot{\phi}}\,, \tag{3.102}$$

where I have used the result of equation (3.92). We have thus obtained a relation between the post-inflation metric perturbation with the inflationary perturbation at horizon crossing. This implies again that the metric power spectrum is given by

$$P_\Phi = P_\Psi = \frac{4}{9}\frac{a^2H^2}{\dot{\phi}^2}|\delta\phi|^2\,. \tag{3.103}$$

Using the result from equation (3.82) we get

$$\begin{aligned} P_\Psi &= \frac{4}{9}\frac{a^2H^2}{\dot{\phi}^2}\frac{H^2}{2k^3}\left(\frac{k}{aH}\right)^{3-2\nu} \\ &= \frac{2}{9k^3}\left(\frac{aH}{\dot{\phi}}\right)^2 H^2\left(\frac{k}{aH}\right)^{3-2\nu} \\ &= \frac{8\pi G}{9k^3}\frac{H^2}{\epsilon}\left(\frac{k}{aH}\right)^{3-2\nu}\,, \end{aligned} \tag{3.104}$$

where in the last line I have used that $(aH/\dot{\phi})^2 = 4\pi G/\epsilon$. We can simplify this expression further by recalling that since $\nu^2 = 9/4 - 3\delta + 3\epsilon$, we get

$$\begin{aligned} \nu &= \frac{3}{2}\sqrt{1 + \left(-\frac{4}{3}\delta + \frac{4}{3}\epsilon\right)} \\ &\simeq \frac{3}{2}\left(1 - \frac{2}{3}\delta + \frac{2}{3}\epsilon\right) \\ &= \frac{3}{2} - \delta + \epsilon\,. \end{aligned} \tag{3.105}$$

Hence the exponent of the k/aH term becomes

$$3 - 2\nu = 2\delta - 2\epsilon\,. \tag{3.106}$$

Defining now the *spectral index* n as $n - 1 \equiv 2\delta - 2\epsilon$ we obtain

$$P_\Psi = \frac{8\pi G}{9k^3}\frac{H^2}{\epsilon}\left(\frac{k}{aH}\right)^{n-1}\,. \tag{3.107}$$

We will need this expression later when we calculate the CMB power spectrum in chapter 6. In our representation of the power spectrum, a spectrum is called *scale invariant* if $k^3 P_\Psi(k)$ is independent of k. This is in fact the case if we require $n = 1$ in equation (3.107). This is equivalent to the well known Harrison-Zel'dovich-Peebles spectrum. We will see in chapter 6 that a scale invariant power spectrum will give rise to a nearly flat Sachs-Wolfe plateau in the CMB spectrum for low multipoles. Recent measurements from the ongoing WMAP experiment seem however to indicate a deviation from the flat spectrum at very large scales.
Recall that we have obtained our result here without making use of the Einstein equations (equations (2.193)-(2.195)). If we had done that equation (3.107) would actually remain unchanged[27]. The difference being in the spectral index which would become

$$n - 1 = 2\delta - 6\epsilon\,. \tag{3.108}$$

Hence our approach gave an error of the order 4ϵ.

3.8.4 Proof of the Constancy of ζ

We will now set about to prove the constancy of ζ, the curvature perturbation function first introduced in section 3.8.1. Recall that it was defined as

$$\zeta = -\frac{ik_i \delta T^0{}_i H}{k^2(\rho + P)} - \Psi\,. \tag{3.109}$$

With a bit of handwaving we used this function to relate the scalar perturbations set up by inflation with the perturbations to the metric.
We begin with the equation of the conservation of energy-momentum given by

$$T^\mu{}_{\nu;\mu} = T^\mu{}_{\nu,\mu} + \Gamma^\mu{}_{\alpha\mu} T^\alpha{}_\nu - \Gamma^\alpha{}_{\nu\mu} T^\mu{}_\alpha = 0\,. \tag{3.110}$$

The metric is now

$$g_{00} = -1 - 2\Psi\,, \qquad g_{ij} = \delta_{ij} a^2 \left(1 + 2\Phi\right)\,. \tag{3.111}$$

We will need the Christoffel symbols. These are

$$\Gamma^0{}_{00} = \frac{\partial \Psi}{\partial t} \tag{3.112}$$

$$\Gamma^0{}_{0i} = \Gamma^0{}_{i0} = \Psi,_i \tag{3.113}$$

$$\Gamma^0{}_{ij} = \delta_{ij} a^2 \left[H + 2H \left(\Phi - \frac{\partial \Psi}{\partial t} \right) \right] \tag{3.114}$$

$$\Gamma^i{}_{00} = \frac{1}{a^2} \Psi,_i \tag{3.115}$$

$$\Gamma^i{}_{j0} = \delta_{ij} \left(H + \frac{\partial \Phi}{\partial t} \right) \tag{3.116}$$

$$\Gamma^i{}_{jk} = \delta_{ij} \Phi,_k + \delta_{ik} \Phi,_j - \delta_{jk} \Phi,_i \ . \tag{3.117}$$

We will now split the perturbed conservation equation into a zero order part and a perturbative part by letting $T^\mu{}_\nu \to T^\mu{}_\nu + \delta T^\mu{}_\nu$. Observe that we will only need the ${}^0{}_0$-component of the energy-momentum equation, and that we as usual will only keep 1. order terms in the perturbations.
We thus get

$$\frac{\partial}{\partial t} \left(T^0_{\ 0} + \delta T^0_{\ 0} \right) + \frac{\partial}{\partial x^i} \left(T^i_{\ 0} + \delta T^i_{\ 0} \right) + \Gamma^\mu{}_{\alpha\mu} \left(T^\alpha_{\ 0} + \delta T^\alpha_{\ 0} \right) - \Gamma^\alpha{}_{0\mu} \left(T^\mu{}_\alpha + \delta T^\mu{}_\alpha \right) = 0 \, . \tag{3.118}$$

Starting with the third term, we obtain

$$\begin{aligned} \Gamma^\mu{}_{\alpha\mu} \left(T^\alpha_{\ 0} + \delta T^\alpha_{\ 0} \right) &= \Gamma^\mu{}_{0\mu} \left(T^0_{\ 0} + \delta T^0_{\ 0} \right) + \Gamma^\mu{}_{i\mu} \left(T^i_{\ 0} + \delta T^i_{\ 0} \right) \\ &= \left[\frac{\partial \Psi}{\partial t} + 3 \left(H + \frac{\partial \Phi}{\partial t} \right) \right] \left(T^0_{\ 0} + \delta T^0_{\ 0} \right) \\ &\quad + \left[\frac{\partial \Psi}{\partial x^i} + \Gamma^i{}_{jk} \right] \left(T^i_{\ 0} + \delta T^i_{\ 0} \right) \ . \end{aligned} \tag{3.119}$$

In the last term, recall that the zero order energy-momentum tensor is diagonal, hence $T^i_{\ 0} = 0$. We are then left with two first order terms multiplying the first order expression $\delta T^i_{\ 0}$. Thus we can set the complete product to 0 in our approximation. We are left with

$$\Gamma^\mu{}_{\alpha\mu} \left(T^\alpha_{\ 0} + \delta T^\alpha_{\ 0} \right) \approx \left[\frac{\partial \Psi}{\partial t} + 3 \left(H + \frac{\partial \Phi}{\partial t} \right) \right] \left(T^0_{\ 0} + \delta T^0_{\ 0} \right) \ . \tag{3.120}$$

Moving now to the last term of equation (3.118), we get

$$\begin{aligned} -\Gamma^\alpha{}_{0\mu} \left(T^\mu{}_\alpha + \delta T^\mu{}_\alpha \right) &= -\Gamma^\alpha{}_{00} \left(T^0{}_\alpha + \delta T^0{}_\alpha \right) - \Gamma^\alpha{}_{0i} \left(T^i{}_\alpha + \delta T^i{}_\alpha \right) \\ &= -\frac{\partial \Psi}{\partial t} \left(T^0_{\ 0} + \delta T^0_{\ 0} \right) - \frac{1}{a^2} \frac{\partial \Psi}{\partial x^i} \left(T^0_{\ i} + \delta T^0_{\ i} \right) \\ &\quad - \Psi,_i \left(T^i_{\ 0} + \delta T^i_{\ 0} \right) - \Gamma^j{}_{0i} \left(T^i_{\ j} + \delta T^i_{\ j} \right) \ . \end{aligned} \tag{3.121}$$

We can as mentioned before ignore all non-diagonal parts of the zero-order energy-momentum tensor (being 0), in addition to all second order terms. We thus obtain

$$\begin{aligned} -\Gamma^{\alpha}{}_{0\mu}\left(T^{\mu}{}_{\alpha}+\delta T^{\mu}{}_{\alpha}\right) &\approx -\frac{\partial\Psi}{\partial t}T^{0}{}_{0}-\Gamma^{j}{}_{0i}\left(T^{i}{}_{j}+\delta T^{i}{}_{j}\right) \\ &= -\frac{\partial\Psi}{\partial t}T^{0}{}_{0}-3HT^{i}{}_{i}-3\frac{\partial\Phi}{\partial t}T^{i}{}_{i}-H\delta T^{i}{}_{i}\,. \end{aligned} \tag{3.122}$$

Here I have used the expression for the Christoffel symbols. Observe that in the last term the sum over i is implicit.
Putting it all together, equation (3.118) becomes

$$\begin{aligned} &\frac{\partial}{\partial t}T^{0}{}_{0}+\frac{\partial}{\partial t}\delta T^{0}{}_{0}+\frac{\partial}{\partial x^{i}}\delta T^{i}{}_{0}+\frac{\partial\Psi}{\partial t}T^{0}{}_{0}+3HT^{0}{}_{0}+3\frac{\partial\Phi}{\partial t}T^{0}{}_{0} \\ &\quad +3H\delta T^{0}{}_{0}-\frac{\partial\Psi}{\partial t}T^{0}{}_{0}-3HT^{i}{}_{i}-3\frac{\partial\Phi}{\partial t}T^{i}{}_{i}-H\delta T^{i}{}_{i}=0\,. \end{aligned} \tag{3.123}$$

Rearrangement gives

$$\begin{aligned} &\frac{\partial}{\partial t}T^{0}{}_{0}+3HT^{0}{}_{0}-3HT^{i}{}_{i}+\frac{\partial}{\partial t}\delta T^{0}{}_{0}+\frac{\partial}{\partial x^{i}}\delta T^{i}{}_{0}+3\frac{\partial\Phi}{\partial t}\left(T^{0}{}_{0}-T^{i}{}_{i}\right) \\ &\qquad +3H\delta T^{0}{}_{0}-H\delta T^{i}{}_{i}=0\,. \end{aligned} \tag{3.124}$$

The first three terms of the above equation add up to 0 by virtue of the zero-order equation $T^{\mu}{}_{\nu;\mu}=0$. It will also be convenient at this stage to return to Fourier space. We are hence left with

$$\frac{\partial}{\partial t}\delta T^{0}{}_{0}+ik_{i}\delta T^{i}{}_{0}+3H\delta T^{0}{}_{0}-H\delta T^{i}{}_{i}=-3\frac{\partial\Phi}{\partial t}\left(T^{0}{}_{0}-T^{i}{}_{i}\right)\,. \tag{3.125}$$

If we now make the following substitutions

$$T^{0}{}_{0}=-\rho,\qquad T^{i}{}_{i}=P,\qquad \Phi=-\Psi\,, \tag{3.126}$$

we get

$$\frac{\partial}{\partial t}\delta T^{0}{}_{0}+ik_{i}\delta T^{i}{}_{0}+3H\delta T^{0}{}_{0}-H\delta T^{i}{}_{i}=-3\frac{\partial\Psi}{\partial t}\left(\rho+P\right)\,. \tag{3.127}$$

Now that we have acquired the conservation equation, let us see where it will lead us. First of all in our setting we are interested in large scales. We can thus omit the term $ik_{i}\delta T^{i}_{0}$, since we already calculated δT^{i}_{0} in section 3.8.2 and discovered that it is of the order k . Hence the term we are neglecting is of the order k^2, which is an acceptable approximation at this level. We are now left with

$$\frac{\partial}{\partial t}\delta T^0_{\ 0} + 3H\delta T^0_{\ 0} - H\delta T^i_{\ i} = -3\frac{\partial \Psi}{\partial t}(\rho + P) \, . \tag{3.128}$$

To go further we will need the perturbed Einstein equations. From [10], these are given by

$$-3H\frac{\partial \Phi}{\partial t} + 3\Psi H^2 - \frac{k^2\Phi}{a^2} = 4\pi G\delta T^0_{\ 0} \tag{3.129}$$

$$ik_i\left(\frac{\partial \Phi}{\partial t} - H\Psi\right) = 4\pi G k_i \delta T^0_{\ i} \, . \tag{3.130}$$

If we contract with k_i and rearrange a little, the last equation becomes

$$3H\frac{\partial \Phi}{\partial t} - 3\Psi H^2 = -\frac{12\pi G i k_i H\delta T^0_{\ i}}{k^2} \, . \tag{3.131}$$

Combining this result with equation (3.129) gives us

$$-\frac{k^2\Phi}{a^2} = 4\pi G\delta T^0_{\ 0} - \frac{12\pi G i k_i H\delta T^0_{\ i}}{k^2} \, . \tag{3.132}$$

Recall that we are working in the large scale regime. We can thus neglect the term on the left. This gives us

$$4\pi G\delta T^0_{\ 0} = \frac{12\pi G i k_i H\delta T^0_{\ i}}{k^2} \tag{3.133}$$

$$\Downarrow$$

$$\frac{ik_i H\delta T^0_{\ i}}{k^2} = \frac{\delta T^0_{\ 0}}{3} \, . \tag{3.134}$$

This enables us to simplify the expression for ζ in the following manner

$$\begin{aligned} \zeta &= -\frac{ik_i H\delta T^0_{\ i}}{k^2(\rho + P)} - \Psi \\ &= -\frac{\delta T^0_{\ 0}}{3(\rho + P)} - \Psi \, . \end{aligned} \tag{3.135}$$

Solving for Ψ and inserting the result into equation (3.128) gives us (working with the right hand side)

$$\begin{aligned} -3\frac{\partial \Psi}{\partial t}(\rho + P) &= 3(\rho + P)\frac{\partial}{\partial t}\left(\zeta + \frac{\delta T^0_{\ 0}}{3(\rho + P)}\right) \\ &= 3(\rho + P)\frac{\partial \zeta}{\partial t} + (\rho + P)\frac{\partial}{\partial t}\left(\frac{\delta T^0_{\ 0}}{\rho + P}\right) \\ &= 3(\rho + P)\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial t}\delta T^0_{\ 0} - \frac{\delta T^0_{\ 0}}{\rho + P}\left(\frac{\partial \rho}{\partial t} + \frac{\partial P}{\partial t}\right) \, . \end{aligned} \tag{3.136}$$

Comparing this result with the left hand side of equation (3.128), we can already see that the $\partial\left(\delta T^0_{\ 0}\right)/\partial t$ term cancel. The full equation is reduced to

$$3H\delta T^0_{\ 0} - H\delta T^i_{\ i} = 3(\rho+P)\frac{\partial\zeta}{\partial t} - \frac{\delta T^0_{\ 0}}{\rho+P}\left(\frac{\partial\rho}{\partial t}+\frac{\partial P}{\partial t}\right). \tag{3.137}$$

Our equation can be simplified even further. The homogenous continuity equation from chapter 1 (equation (1.5)) is given by

$$\frac{\partial\rho}{\partial t} = -3H(\rho+P). \tag{3.138}$$

This gives us

$$3H\delta T^0_{\ 0} - H\delta T^i_{\ i} = 3(\rho+P)\frac{\partial\zeta}{\partial t} + 3H\delta T^0_{\ 0} - \frac{\delta T^0_{\ 0}}{\rho+P}\frac{\partial P}{\partial t} \tag{3.139}$$

$$\Downarrow$$

$$3(\rho+P)\frac{\partial\zeta}{\partial t} = \frac{\delta T^0_{\ 0}}{\rho+P}\frac{dP}{dt} - H\delta T^i_{\ i} \tag{3.140}$$

$$\Downarrow$$

$$\frac{\partial\zeta}{\partial t} = \frac{1}{3(\rho+P)^2}\left(\delta T^0_{\ 0}\frac{dP}{dt} - H(\rho+P)\delta T^i_{\ i}\right). \tag{3.141}$$

We can use equation (3.138) again to obtain

$$\frac{\partial\zeta}{\partial t} = \frac{1}{3(\rho+P)^2}\left(\delta T^0_{\ 0}\frac{dP}{dt} + \frac{\delta T^i_{\ i}}{3}\frac{d\rho}{dt}\right). \tag{3.142}$$

Observe that we can interpret $\delta T^0_{\ 0}$ as the density contrast $-\delta\rho$, and $\delta T^i_{\ i}/3$ as the perturbation to the pressure δP. We can hence write

$$\frac{\partial\zeta}{\partial t} = \frac{1}{3(\rho+P)^2}\left(-\delta\rho\frac{dP}{dt} + \delta P\frac{d\rho}{dt}\right). \tag{3.143}$$

Notice that for adiabatic perturbations we require[10]that

$$\frac{\delta P}{\delta\rho} = c_s^2, \tag{3.144}$$

where c_s is the sound speed defined earlier. An alternative definition [37] of the sound speed is

$$\frac{dP}{d\rho} = c_s^2. \tag{3.145}$$

This implies that $\delta p d\rho = \delta\rho dp$. Inserted into equation (3.143) gives us

$$\frac{\partial\zeta}{\partial t} = 0\,. \tag{3.146}$$

We have thus finally proven that the function ζ is indeed a time independent quantity, hence justifying our previous assumption.
This concludes our work on inflation. Now that we have an understanding of the early universe, we shall return to the Boltzmann equations we acquired in chapter 2 and study these in detail.

Chapter 4

Perturbations in the Cosmic Plasma

4.1 Introduction

In this chapter we will undergo the task of calculating and explaining the dynamics of the CMB power spectrum. We will use the tools devolved in the previous chapter to study this phenomenon in upcoming chapters. The equations describing the CMB spectrum are an infinite set of coupled differential equations, where analytical solutions are impossible to obtain. We will hence make some approximations, specifically the so called *tight coupling limit*, where the baryons and photons are so strongly coupled by Compton collisions that they act as a single fluid[25]. This will simplify matters a great deal, leading us into studying the acoustic oscillations of the cosmic plasma, where the peak locations of these oscillations have direct consequences for the full CMB power spectrum.

4.2 Initial Conditions

We start by dealing with the problem of initial conditions [19]. As we saw in chapter 3, our model for inflation sets an initial non-zero spectrum for Ψ, hence setting the seeds for structure formation. We want to see how this initial excitation of the gravitational potential Ψ effects the temperature perturbation Θ in the large scale [40] limit. Returning for simplicity to cosmic time t, we have that $T \sim a^{-1}$ we have that

$$\Theta \equiv \frac{\delta T}{T} = -\frac{\delta a}{a}. \tag{4.1}$$

Since $a \propto t^{\frac{2}{3(1+\omega)}}$, we obtain

$$\Theta = -\frac{2}{3(1+\omega)} \frac{\delta t}{t}. \tag{4.2}$$

The perturbation $\delta t/t$ can be viewed as a time-dilation effect from the varying gravitational potential Ψ, and since $ds = \sqrt{1+2\Psi}dt \simeq (1+\Psi)dt$, the time perturbation is simply Ψ. We thus obtain

$$\Theta = -\frac{2}{3(1+\omega)} \Psi. \tag{4.3}$$

Since the observed temperature fluctuation is $\Theta + \Psi$ we get

$$\Theta + \Psi = \frac{1+3\omega}{3+3\omega} \Psi. \tag{4.4}$$

In a matter dominated universe with $\omega = 0$ the above equation reduces to $\Theta + \Psi = \frac{1}{3}\Psi$. This is again another manifestation of Sachs-Wolfe effect[40], which we will study more in section 6.4. For now it is suffice to know that inflation sets a non-zero value for the initial perturbations of Θ and Ψ.

4.3 Boltzmann Hierarchy

4.3.1 Multipole Expansion $\tilde{\Theta}_l$

In the previous chapter we saw defined the zeroth and first moment of the temperature perturbation $\Theta(\tilde{\mu}, \eta)$, where $\vec{n} \cdot \vec{k} = \mu/k$, as

$$\tilde{\Theta}_0 = \int_{-1}^{1} \frac{d\mu}{2} \tilde{\Theta}(\mu) \tag{4.5}$$

$$\tilde{\Theta}_1 = i \int_{-1}^{1} \frac{d\mu}{2} \mu \, \tilde{\Theta}(\mu), \tag{4.6}$$

which we refer to as the *monopole* and the *dipole* respectively. We will now generalize this into defining the l'th multipole moment of $\Theta(\tilde{\mu}, \eta)$ as

$$\tilde{\Theta}_l = \frac{1}{(-i)^l} \int_{-1}^{1} \frac{d\mu}{2} P_l(\mu) \, \tilde{\Theta}(\mu), \tag{4.7}$$

where P_l is the l'th order *Legendre polynomial*. The Legendre polynomials are a complete set of orthogonal functions. To name the first polynomials

$$P_0(\mu) = 1 \tag{4.8}$$
$$P_1(\mu) = \mu \tag{4.9}$$
$$P_2(\mu) = \frac{3\mu^2 - 1}{2}. \tag{4.10}$$

An important property of the Legendre polynomials [1] is the recurrence relation

$$(l+1)P_{l+1}(\mu) = (2l+1)\mu P_l(\mu) - lP_{l-1}(\mu). \tag{4.11}$$

We will be using this equation in the next section to obtain the Boltzmann Hierarchy.

4.3.2 Multipole Boltzmann Equations

We will now revisit the full Boltzmann equation obtained in chapter 2 and combine these with the definitions of the previous section. Recall that the photon Boltzmann equation was

$$\dot{\tilde{\Theta}} + i\,k\mu\,\tilde{\Theta} + \dot{\tilde{\Phi}} + i\,k\mu\,\tilde{\Psi} = -\dot{\tau}\big[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\,\tilde{v}_b\big]. \tag{4.12}$$

Observe now that all perturbations that are indexed will have no μ dependance.If we now multiply by $\frac{d\mu}{2}P_0(\mu)$ and integrate, we obtain

$$\begin{aligned}\int_{-1}^{1}\frac{d\mu}{2}\dot{\tilde{\Theta}}+i\int_{-1}^{1}\frac{d\mu}{2}\,k\mu\,\tilde{\Theta}+\int_{-1}^{1}\frac{d\mu}{2}\dot{\tilde{\Phi}}& \\ +\,i\int_{-1}^{1}\frac{d\mu}{2}\,k\mu\,\tilde{\Psi} = -\dot{\tau}\big[\int_{-1}^{1}\frac{d\mu}{2}\tilde{\Theta}_0 - \int_{-1}^{1}\frac{d\mu}{2}\tilde{\Theta} + \int_{-1}^{1}\frac{d\mu}{2}\mu\,\tilde{v}_b\big].&\end{aligned} \tag{4.13}$$

In addition, the gravitational perturbations $\tilde{\Phi}$ and $\tilde{\Psi}$, and the baryon velocity $\tilde{v}_b$, are independent [10] of μ. We then get

$$\dot{\tilde{\Theta}}_0 + k\,\tilde{\Theta}_1 + \dot{\tilde{\Phi}} + 0 = -\dot{\tau}\big[\tilde{\Theta}_0 - \tilde{\Theta}_0 + 0\big]\,, \tag{4.14}$$

which gives

$$\dot{\tilde{\Theta}}_0 + k\,\tilde{\Theta}_1 = -\dot{\tilde{\Phi}} \tag{4.15}$$

Taking now the 2. moment of the Boltzmann equation, i.e. we multiply by $i\frac{d\mu}{2}P_1(\mu)$ and integrate, we get

$$i\int_{-1}^{1}\frac{d\mu}{2}P_1(\mu)\dot{\tilde{\Theta}} - k\int_{-1}^{1}\frac{d\mu}{2}\mu\,P_1(\mu)\tilde{\Theta} + i\int_{-1}^{1}\frac{d\mu}{2}P_1(\mu)\dot{\tilde{\Phi}} - k\int_{-1}^{1}\frac{d\mu}{2}\mu\,P_1(\mu)\tilde{\Psi}$$
$$= -\dot{\tau}\big[i\int_{-1}^{1}\frac{d\mu}{2}P_1(\mu)\tilde{\Theta}_0$$
$$- i\int_{-1}^{1}\frac{d\mu}{2}P_1(\mu)\tilde{\Theta} + i\int_{-1}^{1}\frac{d\mu}{2}\mu\,P_1(\mu)\tilde{v}_b\big]\,. \tag{4.16}$$

The third term on the left hand side and the first term on the right hand side both integrates to 0. In addition we have that

$$\mu P_1(\mu) = \mu^2 = \frac{2}{3}P_2(\mu) + \frac{1}{3}P_0(\mu)\,. \tag{4.17}$$

Using this result we get

$$\dot{\tilde{\Theta}}_1 - k\frac{2}{3}\int_{-1}^{1}\frac{d\mu}{2}P_2(\mu)\tilde{\Theta} - k\frac{1}{3}\int_{-1}^{1}\frac{d\mu}{2}P_0(\mu)\tilde{\Theta} - \frac{k}{3}\tilde{\Psi} = -\dot{\tau}\big[-\tilde{\Theta}_1 + i\frac{\tilde{v}_b}{3}\big]\,, \tag{4.18}$$

which gives

$$\dot{\tilde{\Theta}}_1 - \frac{2}{3}k\tilde{\Theta}_2 - \frac{1}{3}k\tilde{\Theta}_0 - \frac{k}{3}\tilde{\Psi} = \dot{\tau}\big[\tilde{\Theta}_1 - i\frac{\tilde{v}_b}{3}\big]\,. \tag{4.19}$$

Rearrangement leads to

$$\dot{\tilde{\Theta}}_1 - \frac{2k}{3}\tilde{\Theta}_2 - \frac{k}{3}\tilde{\Theta}_0 = \frac{k}{3}\tilde{\Psi} + \dot{\tau}\big[\tilde{\Theta}_1 - i\frac{\tilde{v}_b}{3}\big]\,. \tag{4.20}$$

We can see a pattern emerging, for each moment of the Boltzmann equation we take, we get a coupling of the given multipole with the higher multipole and a lower one. Let us generalize what we have done to all higher moments $l > 2$. We can then multiply by $\frac{1}{(-i)^l}\frac{d\mu}{2}P_l(\mu)$

$$\frac{1}{(-i)^l}\int_{-1}^{1}\frac{d\mu}{2}P_l(\mu)\dot{\tilde{\Theta}} + \frac{k}{(-i)^{l+1}}\int_{-1}^{1}\frac{d\mu}{2}P_l(\mu)\mu\,\tilde{\Theta} + \frac{1}{(-i)^l}\int_{-1}^{1}\frac{d\mu}{2}P_l(\mu)\dot{\tilde{\Phi}} +$$
$$\frac{k}{(-i)^{l+1}}\int_{-1}^{1}\frac{d\mu}{2}P_l(\mu)\mu\,\tilde{\Psi} = -\dot{\tau}\Big[\frac{1}{(-i)^l}\int_{-1}^{1}\frac{d\mu}{2}P_l(\mu)\tilde{\Theta}_0 - \frac{1}{(-i)^l}\int_{-1}^{1}\frac{d\mu}{2}P_l(\mu)\tilde{\Theta}$$
$$+\tilde{v}_b\frac{1}{(-i)^l}\int_{-1}^{1}\frac{d\mu}{2}P_l(\mu)\mu\Big]\,. \tag{4.21}$$

Because of the orthogonality of the $P_l(\mu)$, the two last terms on the left hand side in addition to the first and third term on the right hand side are all 0. The first term on the left hand side is simply $\dot{\tilde{\Theta}}_l$ and the remaining term in the bracket on the right hand side is $-\tilde{\Theta}_l$. We thus obtain

$$\dot{\tilde{\Theta}}_l + \frac{k}{(-i)^{l+1}} \int_{-1}^{1} \frac{d\mu}{2} \mu P_l(\mu)\, \tilde{\Theta} = \dot{\tau} \tilde{\Theta}_l \,. \tag{4.22}$$

From equation (4.11), we have that

$$\mu P_l(\mu) = \frac{l+1}{2l+1} P_{l+1}(\mu) - \frac{l}{2l+1} P_{l-1}(\mu) \,. \tag{4.23}$$

Using this relation, the remaining integral becomes

$$\begin{aligned} \frac{k}{(-i)^{l+1}} \int_{-1}^{1} \frac{d\mu}{2} \mu P_l(\mu)\, \tilde{\Theta} &= k \frac{l+1}{2l+1} \frac{1}{(-i)^{l+1}} \int_{-1}^{1} \frac{d\mu}{2} P_{l+1}(\mu)\, \tilde{\Theta} \\ &\quad - k \frac{l}{2l+1} \frac{1}{(-i)^{l+1}} \int_{-1}^{1} \frac{d\mu}{2} P_{l-1}(\mu)\, \tilde{\Theta} \\ &= k \frac{l+1}{2l+1} \tilde{\Theta}_{l+1} - k \frac{l}{2l+1} \tilde{\Theta}_{l-1} \,. \end{aligned} \tag{4.24}$$

Inserting this into equation (4.22) finally gives

$$\dot{\tilde{\Theta}}_l + k \frac{l+1}{2l+1} \tilde{\Theta}_{l+1} - k \frac{l}{2l+1} \tilde{\Theta}_{l-1} = \dot{\tau} \tilde{\Theta}_l \,. \tag{4.25}$$

This equation, together with equations (4.15) and (4.20), is called the *Boltzmann Hierarchy*. They are an infinite set of coupled ordinary differential equations for the temperature perturbations $\tilde{\Theta}_l$. We will later see that the CMB power spectrum is directly related to these mutltpoles in a relatively simple way, hence the challenge is to calculate the $\tilde{\Theta}_l$. One can use numerical tools to calculate the power spectrum, like the installation *CMBEasy* which uses the *CMBFast* code for the calculations, but has a graphical interface where one can easily vary cosmological parameters. In these codes one obviously has to set a cutoff value for l. This cutoff value must not be set too low because roundoff errors will propagate down to the lower multipoles. The codes typically[28]recommend a cutoff at $l \sim 1500 - 2000$. I will later in chapter 7 use these codes to show some examples of CMB power spectra.

4.4 Tight Coupling Approximation

It is quite hard to obtain analytic solution of the Boltzmann Hierarchy[1] in it's fullness. It is however possible to obtain approximate solutions in certain cosmological eras. Before the time of recombination η_*, the very large amount of free electrons drastically reduces the mean free path of the photons $\sim \eta/\tau$, which corresponds to a very large optical depth $\tau \gg 1$. This implies that the photons behave very much like a fluid, tightly coupled to the electron-proton plasma. Returning to the Boltzmann Hierarchy for $l > 2$, let us make an order of magnitude analysis of this equation. Firstly, the term $\dot{\tilde{\Theta}}_l$ is of the order $\tilde{\Theta}_l/\eta$. Similarly, $\dot{\tau}\tilde{\Theta}_l \sim \tau\tilde{\Theta}_l/\eta$, $k(l+1)/(2l+1)\tilde{\Theta}_{l+1} \sim k\tilde{\Theta}_{l+1}/2$ and $kl/(2l+1)\tilde{\Theta}_{l-1} \sim k\tilde{\Theta}_{l-1}/2$. Hence equation (4.25) becomes

$$\frac{\tilde{\Theta}_l}{\eta} + \frac{k}{2}\tilde{\Theta}_{l+1} - \frac{k}{2}\tilde{\Theta}_{l-1} \sim \frac{\tau}{\eta}\tilde{\Theta}_l \,. \tag{4.26}$$

Since $\tau \gg 1$, we can neglect the first term on the left hand side $\frac{\tilde{\Theta}_l}{\eta}$. We thus get

$$\tilde{\Theta}_{l+1} - \tilde{\Theta}_{l-1} \sim \frac{2\tau}{k\eta}\tilde{\Theta}_l \,. \tag{4.27}$$

The equivalent equation for a higher mutlipole is

$$\tilde{\Theta}_{l+2} - \tilde{\Theta}_l \sim \frac{2\tau}{k\eta}\tilde{\Theta}_{l+1} \,. \tag{4.28}$$

So the order of $\tilde{\Theta}_{l+1}$ is

$$\tilde{\Theta}_{l+1} \sim \frac{k\eta}{2\tau}(\tilde{\Theta}_{l+2} - \tilde{\Theta}_l) \,. \tag{4.29}$$

Inserting this into equation (4.27) gives

$$\frac{k\eta}{2\tau}\tilde{\Theta}_{l+2} - \frac{k\eta}{2\tau}\tilde{\Theta}_l - \tilde{\Theta}_{l-1} \sim \frac{2\tau}{k\eta}\tilde{\Theta}_l \,. \tag{4.30}$$

Horizon sized modes have $k\eta \sim 1$. Hence we can neglect the two first terms on the left hand side. Thus we get

$$\tilde{\Theta}_l \sim \frac{k\eta}{2\tau}\tilde{\Theta}_{l-1} \,. \tag{4.31}$$

Hence in the tightly coupled limit, $|\tilde{\Theta}_l| \ll |\tilde{\Theta}_{l-1}|$. This implies that all higher multipoles other than the monopole and the dipole are highly suppressed.

[1]Impossible ?

Thus we will only need the first two moment equations, omitting $\tilde{\Theta}_2$ in the latter. We have thus gone from a setting with infinitely many equations to a set of three coupled first order equations

$$\dot{\tilde{\Theta}}_0 + k\,\tilde{\Theta}_1 = -\dot{\tilde{\Phi}} \tag{4.32}$$

$$\dot{\tilde{\Theta}}_1 - \frac{k}{3}\tilde{\Theta}_0 = \frac{k}{3}\tilde{\Psi} + \dot{\tau}\big[\tilde{\Theta}_1 - i\frac{\tilde{v}_b}{3}\big] \tag{4.33}$$

$$\tilde{v}_b = -3i\tilde{\Theta}_1 + \frac{R}{\dot{\tau}}\left[\dot{\tilde{v}}_b + \mathcal{H}\,\tilde{v}_b + i\,k\tilde{\Psi}\right], \tag{4.34}$$

where the last equation is the baryon velocity equation obtained in chapter 2. This approach was first formalized in [22].

4.5 Acoustic Oscillation Equation

We want to turn the above equations into one equation for one of the variables. Firstly we can make a simplification in the velocity equation (4.34). The second term is of the order $\sim \tau^{-1}$, hence to lowest order we can set

$$\tilde{v}_b \approx -3i\tilde{\Theta}_1\,. \tag{4.35}$$

Inserting this value into equation (4.34) we get

$$\tilde{v}_b = -3i\tilde{\Theta}_1 + \frac{R}{\dot{\tau}}\left[-3i\dot{\tilde{\Theta}}_1 - 3i\mathcal{H}\,\tilde{\Theta}_1 + i\,k\tilde{\Psi}\right]. \tag{4.36}$$

Combining this expression with equation (4.33) gives

$$\begin{aligned}
\dot{\tilde{\Theta}}_1 - \frac{k}{3}\tilde{\Theta}_0 &= \frac{k}{3}\tilde{\Psi} + \dot{\tau}\Big[\tilde{\Theta}_1 - \frac{i}{3}\left(-3i\tilde{\Theta}_1 + \frac{R}{\dot{\tau}}\left[-3i\dot{\tilde{\Theta}}_1 - 3i\mathcal{H}\,\tilde{\Theta}_1 + i\,k\tilde{\Psi}\right]\right)\Big] \\
&= \frac{k}{3}\tilde{\Psi} + \dot{\tau}\Big[\tilde{\Theta}_1 - \tilde{\Theta}_1 + \frac{R}{\dot{\tau}}\big[\dot{\tilde{\Theta}}_1 + \mathcal{H}\,\tilde{\Theta}_1 - \frac{k\tilde{\Psi}}{3}\big]\Big] \\
&= \frac{k\tilde{\Psi}}{3} - R\dot{\tilde{\Theta}}_1 - R\mathcal{H}\,\tilde{\Theta}_1 + R\frac{k\tilde{\Psi}}{3}\,,
\end{aligned} \tag{4.37}$$

or simply

$$(1+R)\dot{\tilde{\Theta}}_1 + R\mathcal{H}\,\tilde{\Theta}_1 - \frac{k}{3}\tilde{\Theta}_0 = (1+R)\frac{k\tilde{\Psi}}{3}\,. \tag{4.38}$$

Division by $(1+R)$ leads to the following equation

$$\dot{\tilde{\Theta}}_1 + \mathcal{H}\frac{R}{1+R}\,\tilde{\Theta}_1 - \frac{k}{3(1+R)}\tilde{\Theta}_0 = \frac{k\tilde{\Psi}}{3}\,. \tag{4.39}$$

We can now eliminate $\tilde{\Theta}_1$ by rewriting equation (4.32) as

$$\tilde{\Theta}_1 = -\frac{\dot{\tilde{\Theta}}_0}{k} - \frac{\dot{\tilde{\Phi}}}{k} \tag{4.40}$$

$$\dot{\tilde{\Theta}}_1 = -\frac{\ddot{\tilde{\Theta}}_0}{k} - \frac{\ddot{\tilde{\Phi}}}{k}\,. \tag{4.41}$$

Inserting these expressions into equation (4.39) gives

$$-\frac{\ddot{\tilde{\Theta}}_0}{k} - \frac{\ddot{\tilde{\Phi}}}{k} + \mathcal{H}\frac{R}{1+R}\left(-\frac{\dot{\tilde{\Theta}}_0}{k} - \frac{\dot{\tilde{\Phi}}}{k}\right) - \frac{k}{3(1+R)}\tilde{\Theta}_0 = \frac{k\tilde{\Psi}}{3}$$

$$\ddot{\tilde{\Theta}}_0 + \ddot{\tilde{\Phi}} + \mathcal{H}\frac{R}{1+R}\dot{\tilde{\Theta}}_0 + \mathcal{H}\frac{R}{1+R}\dot{\tilde{\Phi}} + \frac{k^2}{3(1+R)}\tilde{\Theta}_0 = -\frac{k^2\tilde{\Psi}}{3}\,. \tag{4.42}$$

Rearrangement leads to

$$\ddot{\tilde{\Theta}}_0 + \mathcal{H}\frac{R}{1+R}\dot{\tilde{\Theta}}_0 + \frac{k^2}{3(1+R)}\tilde{\Theta}_0 = -\ddot{\tilde{\Phi}} - \frac{k^2\tilde{\Psi}}{3} - \mathcal{H}\frac{R}{1+R}\dot{\tilde{\Phi}}\,. \tag{4.43}$$

This is an equation for a damped harmonic oscillator with a driving force $F(k,\eta)$ defined as

$$F(k,\eta) = -\ddot{\tilde{\Phi}} - \frac{k^2\tilde{\Psi}}{3} - \mathcal{H}\frac{R}{1+R}\dot{\tilde{\Phi}} \tag{4.44}$$

The $\dot{\tilde{\Theta}}_0$ can be interpreted as a damping term, thus we can already see that a high baryon number will dampen the oscillations, in addition to the expansion of the background space, quantified by $\mathcal{H}$. We can simplify this term by noting that

$$\begin{aligned}\dot{R} = \frac{d}{d\eta}\left(\frac{3\rho_b}{4\rho_r}\right) = \frac{3}{4}\frac{\dot{\rho}_b\rho_r - \rho_b\dot{\rho}_r}{\rho_r^2} &= \frac{3}{4}\frac{-3\mathcal{H}\rho_b\rho_r + 4\mathcal{H}\rho_b\rho_r}{\rho_r^2} \\ &= \frac{3}{4}\frac{\mathcal{H}\,\rho_b\rho_r}{\rho_r^2} \\ &= \mathcal{H}R\,. \end{aligned} \tag{4.45}$$

We can in addition define the sound speed c_s as

$$c_s^2 \equiv \frac{1}{3(1+R)}\,. \tag{4.46}$$

Using these last results and definitions we finally obtain

$$\ddot{\tilde{\Theta}}_0 + \frac{\dot{R}}{1+R}\dot{\tilde{\Theta}}_0 + k^2c_s^2\tilde{\Theta}_0 = F(k,\eta)\,. \tag{4.47}$$

I will now obtain a semi-analytic solution of this equation by first solving the homogenous counterpart, i.e. with no driving force, utilizing the WKB approximation.

4.6 WKB Solution Of The Homogenous Acoustic Equation

In solving the full oscillator equation (4.47) we will be using a Greens function method consisting of finding the Greens function which is a solution of the homogenous equation[22]. Then we use this function to induce the inhomogeneous solutions which we will do in the next section. Let us first solve the homogenous equation

$$\ddot{\tilde{\Theta}}_0 + \frac{\dot{R}}{1+R}\dot{\tilde{\Theta}}_0 + k^2 c_s^2 \tilde{\Theta}_0 = 0\,. \tag{4.48}$$

Let us now assume a solution of the form

$$\tilde{\Theta}_0 = A e^{iB}\,, \tag{4.49}$$

where A and B are functions of conformal time η. Differentiation gives

$$\dot{\tilde{\Theta}}_0 = \dot{A} e^{iB} + iA\dot{B}e^{iB} \tag{4.50}$$

$$\ddot{\tilde{\Theta}}_0 = \ddot{A} e^{iB} + 2i\dot{A}\dot{B}e^{iB} + iA\ddot{B}e^{iB} - A(\dot{B})^2 e^{iB}\,. \tag{4.51}$$

Inserting this into equation (4.48) gives

$$\ddot{A}e^{iB} + 2i\dot{A}\dot{B}e^{iB} + iA\ddot{B}e^{iB} - A(\dot{B})^2e^{iB} + \frac{\dot{R}}{1+R}\left(\dot{A}e^{iB} + iA\dot{B}e^{iB}\right) + k^2c_s^2Ae^{iB} = 0\,. \tag{4.52}$$

Gathering the real and imaginary parts we get two equations for the unknown functions A and B

$$\ddot{A} + \frac{\dot{R}}{1+R}\dot{A} + k^2c_s^2A - A(\dot{B})^2 = 0 \tag{4.53}$$

$$A\ddot{B} + \frac{\dot{R}}{1+R}A\dot{B} + 2\dot{A}\dot{B} = 0\,. \tag{4.54}$$

The assumption now is that the function B changes much more rapidly than A. We can thus neglect the first two terms in equation (4.53). Assuming that $A \neq 0$, equation (4.53) becomes

$$\begin{aligned} k^2 c_s^2 - (\dot{B})^2 &= 0 \\ \dot{B} &= k\, c_s \end{aligned} \tag{4.55}$$

Integration gives (with $B(0) = 0$)

$$\begin{aligned} \int_0^\eta d\eta' \dot{B} &= k \int_0^\eta d\eta' c_s(\eta') \\ B &= k \int_0^\eta d\eta' c_s(\eta') \equiv k\, r_s(\eta)\,, \end{aligned} \tag{4.56}$$

where we have defined the *sound horizon* $r_s(\eta)$, which I will find an explicit expression for in section 4.9 . We can now use equation (4.54) to find A. First note that

$$\ddot{B}\dot{B} = k^2 c_s \dot{c}_s \tag{4.57}$$

$$\dot{c}_s = -\frac{3}{2} c_s^3 \dot{R}\,. \tag{4.58}$$

Combining this with equation (4.54) we get (after multiplication with $\dot{B}$)

$$\begin{aligned} A\, k^2 c_s \dot{c}_s + \frac{\dot{R}}{1+R} A\, k^2 c_s^2 + 2\dot{A} k^2 c_s^2 &= 0 \\ -\frac{3}{2} A\, c_s^2 \dot{R} + \frac{\dot{R}}{1+R} A + 2\dot{A} &= 0\,. \end{aligned} \tag{4.59}$$

Inserting the expression for the sound speed we obtain

$$\begin{aligned} 3\, c_s^2 \dot{R} - \frac{2\dot{R}}{1+R} &= 4\frac{\dot{A}}{A} \\ -\frac{\dot{R}}{1+R} &= 4\frac{\dot{A}}{A}\,. \end{aligned} \tag{4.60}$$

Integrating on both sides leads to

$$\begin{aligned} 4 \ln A &= -\ln(1+R) \\ A &= (1+R)^{-\frac{1}{4}}\,. \end{aligned} \tag{4.61}$$

Hence the two linearly independent solutions of the homogenous oscillator equation are

$$\tilde{\Theta}_0^1(k,\eta) = (1+R)^{-\frac{1}{4}} \sin(k\, r_s(\eta)) \tag{4.62}$$

$$\tilde{\Theta}_0^2(k,\eta) = (1+R)^{-\frac{1}{4}} \cos(k\, r_s(\eta))\,. \tag{4.63}$$

The solutions are as expected sinusoidal with a varying amplitude $(1+R)^{-\frac{1}{4}}$, where the role of R becomes more apparent as a damping factor, as higher R decreases the amplitude.

4.7 Solving The Inhomogeneous Acoustic Equation

Now that we have a semi-analytic solution of the homogenous equation (4.48), we can embark on solving the full acoustic equation. Before we do that, we can simplify equation (4.47) a little by noting that $\tilde{\Phi}$ appears in almost the same manner as $\tilde{\Theta}_0$. Hence we can write

$$\begin{aligned}\ddot{\tilde{\Theta}}_0 + \ddot{\tilde{\Phi}} + \frac{\dot{R}}{1+R}\dot{\tilde{\Theta}}_0 + \mathcal{H}\frac{R}{1+R}\dot{\tilde{\Phi}} + k^2c_s^2\tilde{\Theta}_0 + k^2c_s^2\tilde{\Phi} &= -\frac{k^2\tilde{\Psi}}{3} + k^2c_s^2\tilde{\Phi} \\ \ddot{\tilde{\Theta}}_0 + \ddot{\tilde{\Phi}} + \frac{\dot{R}}{1+R}\left(\dot{\tilde{\Theta}}_0 + \dot{\tilde{\Phi}}\right) + k^2c_s^2\left(\tilde{\Theta}_0 + \tilde{\Phi}\right) &= \frac{k^2}{3}\left(3\,c_s^2\tilde{\Phi} - \tilde{\Psi}\right)\,,\end{aligned} \tag{4.64}$$

which gives

$$\left[\frac{d^2}{d\eta^2} + \frac{\dot{R}}{1+R}\frac{d}{d\eta} + k^2c_s^2\right]\left[\tilde{\Theta}_0 + \tilde{\Phi}\right] = \frac{k^2}{3}\left[\frac{1}{1+R}\tilde{\Phi} - \tilde{\Psi}\right]\,. \tag{4.65}$$

Note that the homogenous version of this equation is exactly the same as equation (4.48). Hence the solutions found in section 4.6 will be equally applicable for this equation. As a first approximation, we can assume that $R \ll 1$, thus we can omit occurrences of R in the expressions, except in the cosine and sine functions for obvious reasons. Physically this is equivalent to saying that we have very little dampening. We can thus write down the two homogenous solutions of equation (4.65)

$$\tilde{S}_1(k,\eta) = \sin(k\,r_s(\eta)) \tag{4.66}$$

$$\tilde{S}_2(k,\eta) = \cos(k\,r_s(\eta))\,. \tag{4.67}$$

The general solution of equation (4.65) is thus[10]

$$\begin{aligned}\tilde{\Theta}_0(\eta) + \tilde{\Phi}(\eta) =& D_1\tilde{S}_1(k,\eta) + D_2\tilde{S}_2(k,\eta) \\ &+ \frac{k^2}{3}\int_0^\eta d\eta'\left[\tilde{\Phi}(\eta') - \tilde{\Psi}(\eta')\right]\frac{\tilde{S}_1(\eta')\tilde{S}_2(\eta) - \tilde{S}_1(\eta)\tilde{S}_2(\eta')}{\tilde{S}_1(\eta')\dot{\tilde{S}}_2(\eta') - \dot{\tilde{S}}_1(\eta')\tilde{S}_2(\eta')}\,,\end{aligned} \tag{4.68}$$

where D_1 and D_2 are integration constants. Observe that I have set the factor $(1+R)^{-1}$ of the $\tilde{\Phi}$ term to unity.
To find the integration constants we will need knowledge about the initial conditions of the perturbations. For our purposes we will only need to know the main implications of inflation, which is that on early times, the perturbations $\tilde{\Theta}_0(\eta)$ and $\tilde{\Phi}$ are constant and non-zero[10]. This will be enough to determine the constants D_1 and D_2 in this simplified setting. In fact by differentiating equation (4.68) and setting $\eta = 0$ we get

$$\dot{\tilde{\Theta}}_0(0) + \dot{\tilde{\Phi}}(0) = D_1 \,. \tag{4.69}$$

Since the perturbations are constant at early times, $D_1 = 0$. This again implies that $D_2 = \tilde{\Theta}_0(0) + \tilde{\Phi}(0)$. Our solution then becomes

$$\begin{aligned}\tilde{\Theta}_0(\eta) + \tilde{\Phi}(\eta) = &\left[\tilde{\Theta}_0(0) + \tilde{\Phi}(0)\right]\cos(k\,r_s(\eta)) \\ &+ \frac{k^2}{3}\int_0^\eta d\eta' \left[\tilde{\Phi}(\eta') - \tilde{\Psi}(\eta')\right] \frac{\tilde{S}_1(\eta')\tilde{S}_2(\eta) - \tilde{S}_1(\eta)\tilde{S}_2(\eta')}{\tilde{S}_1(\eta')\dot{\tilde{S}}_2(\eta') - \dot{\tilde{S}}_1(\eta')\tilde{S}_2(\eta')} \,.\end{aligned} \tag{4.70}$$

In addition we have that

$$\begin{aligned}\tilde{S}_1(\eta')\tilde{S}_2(\eta) - \tilde{S}_1(\eta)\tilde{S}_2(\eta') &= \sin(k\,r_s(\eta'))\cos(k\,r_s(\eta)) \\ &\quad - \sin(k\,r_s(\eta))\cos(k\,r_s(\eta')) \\ &= -\sin\left(k(r_s - r_s')\right) \,,\end{aligned} \tag{4.71}$$

and the denominator in the integral is

$$\begin{aligned}\tilde{S}_1(\eta')\dot{\tilde{S}}_2(\eta') - \dot{\tilde{S}}_1(\eta')\tilde{S}_2(\eta') &= -kc_s(\eta')\sin^2(kr_s') - kc_s(\eta')\cos^2(kr_s') \\ &= -kc_s(\eta') \\ &= -k\frac{1}{\sqrt{3(1+R')}} \\ &\approx -\frac{k}{\sqrt{3}} \,.\end{aligned} \tag{4.72}$$

Using these results we obtain

$$\begin{aligned}\tilde{\Theta}_0(\eta) + \tilde{\Phi}(\eta) = &\left[\tilde{\Theta}_0(0) + \tilde{\Phi}(0)\right]\cos(k\,r_s(\eta)) \\ &+ \frac{k}{\sqrt{3}}\int_0^\eta d\eta' \left[\tilde{\Phi}(\eta') - \tilde{\Psi}(\eta')\right]\sin\left(k(r_s - r_s')\right) \,.\end{aligned} \tag{4.73}$$

We can now commence in finding the dipole $\tilde{\Theta}_1$. Recall that equation (4.32) says that

$$k\tilde{\Theta}_1 = -\left(\dot{\tilde{\Theta}}_0 + \dot{\tilde{\Phi}}\right) . \tag{4.74}$$

Differentiating equation (4.73) with respect to η leads to

$$\begin{aligned}\dot{\tilde{\Theta}}_0(\eta) + \dot{\tilde{\Phi}}(\eta) =& - \left[\tilde{\Theta}_0(0) + \tilde{\Phi}(0)\right] k\, c_s \sin(k\, r_s(\eta)) \\ &+ \frac{k}{\sqrt{3}} k\, c_s \int_0^\eta d\eta' \left[\tilde{\Phi}(\eta') - \tilde{\Psi}(\eta')\right] \cos\left(k(r_s - r_s')\right) \\ \approx& - \frac{k}{\sqrt{3}} \left[\tilde{\Theta}_0(0) + \tilde{\Phi}(0)\right] \sin(k\, r_s(\eta)) \\ &+ \frac{k^2}{3} \int_0^\eta d\eta' \left[\tilde{\Phi}(\eta') - \tilde{\Psi}(\eta')\right] \cos\left(k(r_s - r_s')\right) .\end{aligned} \tag{4.75}$$

Then the dipole becomes

$$\begin{aligned}\tilde{\Theta}_1(\eta) =& \frac{1}{\sqrt{3}} \left[\tilde{\Theta}_0(0) + \tilde{\Phi}(0)\right] \sin(k\, r_s(\eta)) \\ &- \frac{k}{3} \int_0^\eta d\eta' \left[\tilde{\Phi}(\eta') - \tilde{\Psi}(\eta')\right] \cos\left(k(r_s - r_s')\right) .\end{aligned} \tag{4.76}$$

We will analyze these results in section 4.10, specifically about the location of the peaks in the CMB spectrum. Let us now look at a specific case of the acoustic equation where we can obtain a semi-analytic solution.

4.8 Inhomogeneous Solution for a Constant $\tilde{\Phi}$

We can now simplify our solution somewhat by going back to our acoustic equation (equation (4.65)) and assume from the beginning that the gravitational potential is approximately constant. The time variation of the potential $\tilde{\Phi}$ is quite small compared to the monopole and dipole in the matter dominated era [10], hence we will make this approximation at this time. We will thus commence in finding a solution for the dipole and the monopole in this setting. Working with the monopole, recall that the WKB solution of the homogenous equation was

$$[\tilde{\Theta}_0(\eta) + \tilde{\Phi}]_{\text{Hom}} = (1+R)^{-\frac{1}{4}}\, \mathcal{C}_A \cos k r_s . \tag{4.77}$$

where $\mathcal{C}_A$ is an integration constant. Since the right hand side of equation (4.65) is now taken to be constant, we can find a solution by adding an unknown constant to our expression above, i.e

$$[\tilde{\Theta}_0(\eta) + \tilde{\Phi}] = [\tilde{\Theta}_0(\eta) + \tilde{\Phi}]_{\text{Hom}} + A(k). \tag{4.78}$$

Inserting our above guess into equation (4.65) will lead to a cancelation of the homogenous solution. We are left with

$$k^2 c_s^2 A(k) = \frac{k^2}{3}\left[\frac{1}{1+R}\tilde{\Phi} - \tilde{\Psi}\right]. \tag{4.79}$$

Recalling that $c_s^2 = 1/(3(1+R))$ we obtain

$$A(k) = \tilde{\Phi} - (1+R)\tilde{\Psi}. \tag{4.80}$$

We can hence write down the full solution

$$[\tilde{\Theta}_0(\eta) + \tilde{\Phi}] = (1+R)^{-\frac{1}{4}}\,\mathcal{C}_A \cos kr_s + \tilde{\Phi} - (1+R)\tilde{\Psi}, \tag{4.81}$$

which we can simplify as

$$\tilde{\Theta}_0(\eta) = (1+R)^{-\frac{1}{4}}\,\mathcal{C}_A \cos kr_s - (1+R)\tilde{\Psi}. \tag{4.82}$$

This form is quite satisfactory as it actually tells us a great deal about the nature of the perturbations. First we can observe that the zero point of the oscillations is displaced by an amount of $(1+R)\tilde{\Psi}$. Since the sound speed is given by $c_s^2 = 1/(3(1+R))$, the effective mass of our oscillator is approximately $(1+R)$. Hence an increase in R increases the effective inertia, dampening the oscillations, which one can see from the appearance of the $(1+R)^{-\frac{1}{4}}$ factor. For the dipole we obtain

$$\tilde{\Theta}_1(\eta) = \frac{(1+R)^{-\frac{3}{4}}}{\sqrt{3}}\mathcal{C}_A \sin kr_s\,. \tag{4.83}$$

We will in chapter 6 return to these expressions when studying the small scale temperature anistropies.

4.9 The Sound Horizon

We defined the sound horizon as

$$r_s(\eta) = \frac{1}{\sqrt{3}}\int_0^{\eta} d\eta' \frac{1}{\sqrt{1+R'}}\,, \tag{4.84}$$

where all the primes from now on denote quantities that are functions of η'. If the sound speed is constant, the sound horizon is simply $r_s = \eta c_s$. This is not a very realistic scenario, requiring either the baryons to be completely dominant, or that the baryon to photon ratio is constant, which in fact should be of the order $\sim a$. In the tightly coupled limit, the universe is dominated by both matter and radiation, so I will take this assumption to hold here. I will however make no presumption on whether matter-radiation equality has occurred or not. It is however generally assumed that recombination occurs long after the era of equality η_{eq} . I will now calculate an expression for r_s in this setting. We will first need the expansion factor for a matter-radiation dominated universe. This is given by

$$\eta = \frac{2}{\sqrt{\Omega_m H_0^2}} \left[\sqrt{a + a_{eq}} - \sqrt{a_{eq}} \right] , \tag{4.85}$$

where $a_{eq} = a(\eta_{eq})$, H_0 is the Hubble rate presently and Ω_m is the matter density. One way to calculate the integral in equation (4.84) is to change the integration measure $d\eta'$ to dR'. This can be accomplished by recalling that in section 4.5 we proved that

$$\dot{R} = \mathcal{H} R \,. \tag{4.86}$$

Hence

$$dR = \frac{da}{a} R \,. \tag{4.87}$$

This implies that

$$\begin{aligned} \ln R &= \ln a + c \\ R &= \lambda\, a \end{aligned} \tag{4.88}$$

where λ is a constant which we could in principle find, but we will not actually need it. This result is of course expected since $\rho_b \sim a^{-3}$ and $\rho_r \sim a^{-4}$. Returning to equation (4.85) which after differentiating becomes

$$d\eta = \frac{1}{\sqrt{\Omega_m H_0^2}} \frac{da}{\sqrt{a + a_{eq}}} \,, \tag{4.89}$$

which when combined with equation (4.87) gives

$$\begin{aligned} dR &= \frac{da}{a} R = \lambda\, da \\ &= \lambda \sqrt{\Omega_m H_0^2} \sqrt{a + a_{eq}}\, d\eta \\ &= \sqrt{\lambda \Omega_m H_0^2} \sqrt{R + R_{eq}}\, d\eta \,, \end{aligned} \tag{4.90}$$

where I have defined $R_{eq} = R(\eta_{eq})$. Hence we obtain

$$d\eta = \frac{1}{\sqrt{\lambda \Omega_m H_0^2}} \frac{dR}{\sqrt{R + R_{eq}}} \,. \tag{4.91}$$

Thus the expression for the sound horizon becomes

$$r_s(\eta) = \frac{1}{\sqrt{3\,\lambda \Omega_m H_0^2}} \int_0^R dR' \frac{1}{\sqrt{R' + R_{eq}}\sqrt{1 + R'}} \,. \tag{4.92}$$

This integral is analytically solvable. Performing the change of variable

$$u = \ln\left[\sqrt{R' + R_{eq}} + \sqrt{1 + R'}\right] \tag{4.93}$$

$$2\,du = \frac{dR}{\sqrt{R' + R_{eq}}\sqrt{1 + R'}} \,, \tag{4.94}$$

we get

$$\begin{aligned} r_s(\eta) &= \frac{2}{\sqrt{3\,\lambda \Omega_m H_0^2}} \int_{u(0)}^{u(R)} du \\ &= \frac{2}{\sqrt{3\,\lambda \Omega_m H_0^2}} \left[u(R) - u(0)\right] \\ &= \frac{2}{\sqrt{3\,\lambda \Omega_m H_0^2}} \ln\left[\frac{\sqrt{R + R_{eq}} + \sqrt{1 + R}}{1 + \sqrt{R_{eq}}}\right] \,. \end{aligned} \tag{4.95}$$

We can make additional adjustments to the above expression. For reasons that will become clearer later, it is convenient to define the wavenumber of the mode k_{eq} which equals the comoving Hubble radius at matter-radiation equality, i.e when $k_{eq} = a_{eq} H(a_{eq})$. The Hubble factor at equality is

$$H^2(\eta_{eq}) = H_0^2\, \Omega_{tot} = \frac{2 H_0^2\, \Omega_m}{a_{eq}^3} \,. \tag{4.96}$$

This gives

$$k_{eq} = a_{eq} H(a_{eq}) = a_{eq} \sqrt{\frac{2 H_0^2 \Omega_m}{a_{eq}^3}} = \sqrt{\frac{2 H_0^2 \Omega_m}{a_{eq}}} \,. \tag{4.97}$$

The above definition leads to

$$\frac{1}{2} \lambda\, a_{eq} k_{eq}^2 = \lambda\, H_0^2 \Omega_m \,, \tag{4.98}$$

which gives

$$r_s(\eta) = \frac{2\sqrt{2}}{k_{eq}\sqrt{3\,\lambda a_{eq}}} \ln\left[\frac{\sqrt{R+R_{eq}}+\sqrt{1+R}}{1+\sqrt{R_{eq}}}\right] . \tag{4.99}$$

Thus the sound horizon of the baryon-photon fluid becomes

$$r_s(\eta) = \frac{2}{3\,k_{eq}}\sqrt{\frac{6}{R_{eq}}} \ln\left[\frac{\sqrt{R+R_{eq}}+\sqrt{1+R}}{1+\sqrt{R_{eq}}}\right] . \tag{4.100}$$

We will see in chapter 7 how the changing some cosmological parameters, mainly Ω_{m0} and Ω_b, effects the sound horizon.

4.10 The Acoustic Peaks

Let us now return to our description of the photon monopole and dipole and see what information we can obtain on the acoustic peaks. Observe that have neglected diffusion damping effects in the plasma for time being. First if we omit the integrals in the expressions we are left with

$$\tilde{\Theta}_0(\eta) + \tilde{\Phi}(\eta) = \left[\tilde{\Theta}_0(0) + \tilde{\Phi}(0)\right] \cos(k\, r_s(\eta)) \tag{4.101}$$

$$\tilde{\Theta}_1(\eta) = \frac{k}{\sqrt{3}} \left[\tilde{\Theta}_0(0) + \tilde{\Phi}(0)\right] \sin(k\, r_s(\eta))\,. \tag{4.102}$$

Already we can spot the locations of the peaks. For the monopole these are

$$k_0 = \frac{n\pi}{r_s} \qquad n = 0, 1, 2 \ldots. \tag{4.103}$$

and for the dipole

$$k_1 = \left(n + \frac{1}{2}\right)\frac{\pi}{2r_s} \qquad n = 0, 1, 2 \ldots. \tag{4.104}$$

Observe that the dipole and monopole are out of phase. This will obviously have an effect on the final CMB power spectrum as higher multipoles will be out of phase with the lower ones. As we will see the peak location of the monopole is the most significant one, the higher multipoles being suppressed [22]. If we in addition include the effects of gravity through Ψ as we did in section 4.8, there will be an alteration in the hight of the peaks [20]. We will study more of these effects in chapter 6 where we finally introduce the CMB power spectrum.

4.11 Free Streaming Solution

We have up to now seen how in the tight-coupling limit the perturbations in the cosmological plasma induce acoustic oscillations that have well defined peaks. The peaks of the monopole $\tilde{\Theta}_0$ will as we will see correspond to the peaks in the total CMB anisotropy spectrum. As the universe expands cooling will naturally occur. The photon-baryon plasma will be tightly coupled as photon mean free paths are quite low $\sim \tau^{-1}$ the temperature reaches $T \sim 6 - 7000\,\mathrm{k}$. At this stage stable hydrogen can exist, and we have reached the era of recombination η_*. The acoustic state which the universe was in at this time will then have been frozen into the radiation emitted. The photon mean free paths drastically increase, the universe becomes transparent.

In this section we want to study how the perturbations $\tilde{\Theta}_l$ evolve as the photons free stream towards us from the last scattering surface. In section 2.1.8 we already calculated the effects of free streaming in the absence of collisions with other particles. Though this approach is inherently flawed, it hinted towards what effects we might expect, as we encountered the Sachs-Wolfe effect and the integrated Sachs-Wolfe effect for the first time. We will thus find an expression for the perturbations at our time η_0 , using a similar procedure as in section 2.1.8, bearing in mind that scattering effects will complicate matters to a certain extent. First let us introduce a function that will turn out to be quite useful in our calculations.

4.11.1 The Visibility Function

In the cosmic plasma, photons are continuously colliding with other particles, creating an effective pressure effect. The mean free path of the photons is quite small, but as recombination occurs the photons can travel further and further before encountering other particles. We will now define[10] the *visibility function* as

$$g(\eta) \equiv -\dot{\tau} e^{-\tau}, \tag{4.105}$$

where τ is the optical depth introduced earlier. Recall that at early times, τ is very large. Conversely, τ is very small at $\eta = \eta_0$. This implies that

$$\int_0^{\eta_0} g(\eta) d\eta = -\int_0^{\eta_0} \dot{\tau} e^{-\tau} d\eta = \left[e^{-\tau}\right]_0^{\eta_0} = 1. \tag{4.106}$$

Thus g can be interpreted as a probability. It is the probability that an observed photon *last* scattered at time η, thereby retaining the information of it's last scattering as it free streams towards us. In [10] , Dodelsen shows that g is quite small for times prior and after recombination. The reason for this is that τ is quite large at early times, and that $-\dot{\tau} = \sigma n_e a$ decreases rapidly after η_* . In fact g is strongly peaked[25] about $\eta = \eta_*$, at least compared to $\tilde{\Theta}_0(\eta)$ and $\tilde{\Phi}(\eta)$. We will use this to our advantage in the calculations in the next section.

4.11.2 Calculation of $\tilde{\Theta}_l(\eta_0)$

We will now undertake the task of finding out hoe the perturbations at early times evolve as the photons free stream towards us. For this we return to the Boltzmann equation which was

$$\dot{\tilde{\Theta}} + i\,k\mu\,\tilde{\Theta} + \dot{\tilde{\Phi}} + i\,k\mu\,\tilde{\Psi} = -\dot{\tau}\big[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\,\tilde{v}_b\big]. \tag{4.107}$$

We can rewrite the above equation as

$$\dot{\tilde{\Theta}} + (i\,k\mu\, - \dot{\tau})\,\tilde{\Theta} = -\dot{\tilde{\Phi}} - ik\mu\,\tilde{\Psi} - \dot{\tau}\big[\tilde{\Theta}_0 + \mu\,\tilde{v}_b\big] \equiv S, \tag{4.108}$$

where we have defined the right hand side as a source term S with

$$S = -\dot{\tilde{\Phi}} - ik\mu\,\tilde{\Psi} - \dot{\tau}\big[\tilde{\Theta}_0 + \mu\,\tilde{v}_b\big]. \tag{4.109}$$

The left hand side of equation (4.108) we can write

$$e^{-ik\mu\eta+\tau}\frac{\partial}{\partial\eta}\left(\tilde{\Theta}e^{ik\mu\eta-\tau}\right), \tag{4.110}$$

as we did in the collisionless case. Equation (4.108) thus becomes

$$\frac{\partial}{\partial\eta}\left(\tilde{\Theta}e^{ik\mu\eta-\tau}\right) = e^{ik\mu\eta-\tau}S. \tag{4.111}$$

Integration from $\eta = 0$ to $\eta = \eta_0$ gives

$$\begin{aligned}\left[\tilde{\Theta}e^{ik\mu\eta-\tau}\right]_0^{\eta_0} &= \int_0^{\eta_0} d\eta e^{ik\mu\eta-\tau}S \\ \tilde{\Theta}(\eta_0)e^{ik\mu\eta_0-\tau(\eta_0)} - \tilde{\Theta}(0)e^{-\tau(0)} &= \int_0^{\eta_0} d\eta e^{ik\mu\eta-\tau}S.\end{aligned} \tag{4.112}$$

Recall that $\tau(0)$ is very large, hence the second term on the left hand side tends to 0. In addition $\tau(\eta_0)$ is completely negligible. We thus obtain

$$\tilde{\Theta}(\eta_0)e^{ik\mu\eta_0} = \int_0^{\eta_0} d\eta e^{ik\mu\eta-\tau}S, \tag{4.113}$$

which when multiplied by $e^{-ik\mu\eta_0}$ leads to

$$\tilde{\Theta}(\eta_0) = \int_0^{\eta_0} d\eta e^{ik\mu(\eta-\eta_0)-\tau} S. \tag{4.114}$$

We would at this stage like to find an expression for the multipoles $\tilde{\Theta}_l$. To obtain these we need to multiply by $d\mu P_l(\mu)/2(-i)^l$ and integrate over all μ, i.e. from $\mu = -1$ to $\mu = 1$. If we do this to equation (4.114) , the left hand side is trivial, it is simply $\tilde{\Theta}_l$. But the right hand side is a completely different matter, as the source function S has an explicit μ dependence where our formulas of Legendre functions do not seem to apply. We can circumvent this problem by noting that for a function $f = f(\eta)$

$$\begin{aligned}
\int_0^{\eta_0} d\eta\, \mu e^{ik\mu(\eta-\eta_0)-\tau} f &= \int_0^{\eta_0} d\eta\, f \frac{e^{-\tau}}{ik} \frac{\partial}{\partial\eta}\left(e^{ik\mu(\eta-\eta_0)}\right) \\
&= \left[f \frac{e^{-\tau}}{ik} e^{ik\mu(\eta-\eta_0)} \right]_0^{\eta_0} - \frac{1}{ik}\int_0^{\eta_0} d\eta\, e^{ik\mu(\eta-\eta_0)} \frac{\partial}{\partial\eta}\left(fe^{-\tau}\right) \\
&= f(\eta_0)\frac{e^{-\tau(\eta_0)}}{ik} - \frac{1}{ik}\int_0^{\eta_0} d\eta\, e^{ik\mu(\eta-\eta_0)} \frac{\partial}{\partial\eta}\left(fe^{-\tau}\right) \\
&= \frac{f(\eta_0)}{ik} - \frac{1}{ik}\int_0^{\eta_0} d\eta\, e^{ik\mu(\eta-\eta_0)} \frac{\partial}{\partial\eta}\left(fe^{-\tau}\right).
\end{aligned} \tag{4.115}$$

The first term here has no μ dependence. It will become 0 as we integrate over μ for all $l > 0$. Only $l = 0$ will be non-zero, but this will contribute to the only non-measurable part of the perturbations, namely $\tilde{\Theta}_0$. We can hence omit the first term[10]. The above equation then becomes

$$\int_0^{\eta_0} d\eta\, \mu e^{ik\mu(\eta-\eta_0)-\tau} f = -\frac{1}{ik}\int_0^{\eta_0} d\eta\, e^{ik\mu(\eta-\eta_0)} \frac{\partial}{\partial\eta}\left(fe^{-\tau}\right). \tag{4.116}$$

We see that each occurrence of μ can be replaced by $-\frac{1}{ik}\frac{\partial}{\partial\eta}$ in our calculations. Returning to equation (4.114), the right hand side becomes

$$\begin{aligned}
\int_0^{\eta_0} d\eta e^{ik\mu(\eta-\eta_0)-\tau} S &= \int_0^{\eta_0} d\eta(-\dot{\tilde{\Phi}} - ik\mu\, \tilde{\Psi} - \dot{\tau}\left[\tilde{\Theta}_0 + \mu\, \tilde{v}_b\right])e^{ik\mu(\eta-\eta_0)-\tau} \\
&= \int_0^{\eta_0} d\eta \Big[(-\dot{\tilde{\Phi}} - \dot{\tau}\tilde{\Theta}_0)e^{-\tau} e^{ik\mu(\eta-\eta_0)} \\
&\qquad - \left[ik\mu\, \tilde{\Psi} e^{-\tau} + \dot{\tau}\mu\, \tilde{v}_b\, e^{-\tau}\right] e^{ik\mu(\eta-\eta_0)}\Big] \\
&= \int_0^{\eta_0} d\eta \Big[(-\dot{\tilde{\Phi}} - \dot{\tau}\tilde{\Theta}_0)e^{-\tau} e^{ik\mu(\eta-\eta_0)} \\
&\qquad + e^{ik\mu(\eta-\eta_0)}\frac{\partial}{\partial\eta}\left(\tilde{\Psi} e^{-\tau} + \frac{\dot{\tau}}{k} i\tilde{v}_b\, e^{-\tau}\right)\Big].
\end{aligned} \tag{4.117}$$

It is now appropriate to insert the visibility function $g = -\dot{\tau}e^{-\tau}$, introduced in section 4.11.1. This gives

$$\begin{aligned}\tilde{\Theta}(\eta_0) &= \int_0^{\eta_0} d\eta \Big[-\dot{\tilde{\Phi}}e^{-\tau} + g\tilde{\Theta}_0 + \frac{\partial}{\partial\eta}\left(\tilde{\Psi}e^{-\tau} + \frac{i\tilde{v}_b g}{k}\right)\Big] e^{ik\mu(\eta-\eta_0)} \\ &= \int_0^{\eta_0} d\eta \Big[-\dot{\tilde{\Phi}}e^{-\tau} + g\tilde{\Theta}_0 + \dot{\tilde{\Psi}}e^{-\tau} - \tilde{\Psi}\dot{\tau}e^{-\tau} + \frac{\partial}{\partial\eta}\left(\frac{i\tilde{v}_b g}{k}\right)\Big] e^{ik\mu(\eta-\eta_0)} \\ &\equiv \int_0^{\eta_0} d\eta \bar{S} e^{ik\mu(\eta-\eta_0)}, \end{aligned} \tag{4.118}$$

where we have conveniently set

$$\bar{S} = \left(\dot{\tilde{\Psi}} - \dot{\tilde{\Phi}}\right) e^{-\tau} + \left[\tilde{\Theta}_0 + \tilde{\Psi}\right] g + \frac{\partial}{\partial\eta}\left(\frac{i\,\tilde{v}_b g}{k}\right). \tag{4.119}$$

As one can see, we have with the above calculations effectively removed the μ-factor from the integrand. We are now ready to proceed with the task of finding $\tilde{\Theta}_l$. We have now that

$$\tilde{\Theta}(\eta_0) = \int_0^{\eta_0} d\eta \bar{S} e^{ik\mu(\eta-\eta_0)}. \tag{4.120}$$

Multiplying by $d\mu P_l(\mu)/2$ and integrating gives

$$\int_{-1}^{1} \frac{d\mu}{2} P_l(\mu)\tilde{\Theta}(\eta_0) = \int_0^{\eta_0} d\eta \bar{S} \int_{-1}^{1} \frac{d\mu}{2} P_l(\mu) e^{ik\mu(\eta-\eta_0)} \tag{4.121}$$

$$\Downarrow$$

$$(-i)^l \tilde{\Theta}_l(\eta_0) = \int_0^{\eta_0} d\eta \bar{S} \int_{-1}^{1} \frac{d\mu}{2} P_l(\mu) e^{ik\mu(\eta-\eta_0)}. \tag{4.122}$$

It turns out that we can use the our knowledge on spherical Bessel functions to calculate the μ-integral on the right hand side [1] , as we have that $\int_{-1}^{1} d\mu P_l(\mu) e^{ix\mu} = 2j_l(x)/(-i)^l$, where $j_l(x)$ is the l'th spherical Bessel function. We thus obtain

$$\tilde{\Theta}_l(\eta_0) = (-1)^l \int_0^{\eta_0} d\eta\, \bar{S} j_l\left(k(\eta - \eta_0)\right), \tag{4.123}$$

where I have used that $(-i)^l(-i)^l = (-i)^{2l} = (-1)^l$. Taking in addition, the formula $(-1)^l j_l(x) = j_l(-x)$, into account the above equation reduces to

$$\tilde{\Theta}_l(\eta_0) = \int_0^{\eta_0} d\eta\, \bar{S} j_l\left(k(\eta_0 - \eta)\right). \tag{4.124}$$

Let us now write out the full expression of the integral. We find that

$$\begin{aligned}\tilde{\Theta}_l(\eta_0) = &\int_0^{\eta_0} d\eta\, g(\eta)\left[\tilde{\Theta}_0 + \tilde{\Psi}\right] j_l\left(k(\eta_0-\eta)\right)\\ &+\int_0^{\eta_0} d\eta\,\frac{\partial}{\partial\eta}\left(\frac{i\,\tilde{v}_b g}{k}\right) j_l\left(k(\eta_0-\eta)\right)\\ &+\int_0^{\eta_0} d\eta\, e^{-\tau}\left(\dot{\tilde{\Psi}}-\dot{\tilde{\Phi}}\right) j_l\left(k(\eta_0-\eta)\right).\end{aligned} \tag{4.125}$$

It is at this stage where the visibility function comes in really handy. Because of the strongly peaked nature [25], at least compared with $\tilde{\Psi}$ and $\tilde{\Phi}$, and the fact that prior and after it's peak g is effectively 0. The function g will in fact mimic a delta function behavior centered about $\eta = \eta_*$. The integrals involving g will be greatly simplified. The first integral is simply

$$\begin{aligned}\int_0^{\eta_0} d\eta\, g(\eta)\left[\tilde{\Theta}_0(\eta)+\tilde{\Psi}(\eta)\right] j_l\left(k(\eta_0-\eta)\right) \simeq &\left[\tilde{\Theta}_0(\eta_*)+\tilde{\Psi}(\eta_*)\right]\\ &\times j_l\left(k(\eta_0-\eta_*)\right).\end{aligned} \tag{4.126}$$

The second integral of equation (4.125) will require a little more work, but not much. Firstly, integration by parts gives (recall that $g(0) = g(\eta_0) = 0$)

$$\begin{aligned}\int_0^{\eta_0} d\eta\,\frac{\partial}{\partial\eta}\left(\frac{i\,\tilde{v}_b g}{k}\right) j_l\left(k(\eta_0-\eta)\right) &= \left[\frac{i\,\tilde{v}_b g(\eta)}{k} j_l\left(k(\eta_0-\eta)\right)\right]_0^{\eta_0}\\ &\quad -\int_0^{\eta_0} d\eta\,\frac{i\,\tilde{v}_b g(\eta)}{k}\frac{\partial}{\partial\eta} j_l\left(k(\eta_0-\eta)\right)\\ &= -\int_0^{\eta_0} d\eta\,\frac{i\,\tilde{v}_b g(\eta)}{k}\frac{\partial}{\partial\eta} j_l\left(k(\eta_0-\eta)\right)\end{aligned} \tag{4.127}$$

Using that $\frac{dj_l}{dx} = j_{l-1} - \frac{l+1}{x} j_l$ with $x = k(\eta - \eta_*)$ gives

$$\begin{aligned}\int_0^{\eta_0} d\eta\,\frac{\partial}{\partial\eta}\left(\frac{i\,\tilde{v}_b g}{k}\right) j_l\left(k(\eta_0-\eta)\right) &= -\int_0^{\eta_0} d\eta\,\frac{i\,\tilde{v}_b g(\eta)}{k}\Big[j_{l-1}\left(k(\eta_0-\eta)\right)\\ &\qquad -\frac{l+1}{k(\eta_0-\eta)} j_l\left(k(\eta_0-\eta)\right)\Big]\\ &\simeq i\,\tilde{v}_b(\eta_*)\Big[j_{l-1}\left(k(\eta_0-\eta_*)\right)\\ &\qquad -\frac{l+1}{k(\eta_0-\eta_*)} j_l\left(k(\eta_0-\eta)\right)\Big].\end{aligned} \tag{4.128}$$

At recombination $\dot{\tau}$ is very large [10]. We can hence use as we did before that $v_b \simeq -3i\tilde{\Theta}_1$ in the above expression. Putting it all together we finally obtain

$$\begin{aligned}\tilde{\Theta}_l(\eta_0) &= \left[\tilde{\Theta}_0(\eta_*) + \tilde{\Psi}(\eta_*)\right] j_l\left(k(\eta_0 - \eta_*)\right) \\ &+ 3\tilde{\Theta}_1(\eta_*)\left[j_{l-1}\left(k(\eta_0 - \eta_*)\right) - \frac{l+1}{k(\eta_0 - \eta_*)} j_l\left(k(\eta_0 - \eta)\right)\right] \\ &+ \int_0^{\eta_0} d\eta\, e^{-\tau}\left(\dot{\tilde{\Psi}} - \dot{\tilde{\Phi}}\right) j_l\left(k(\eta_0 - \eta)\right). \end{aligned} \tag{4.129}$$

This equation is a very important to our work and is the basis for the CMB power spectrum we are going to study in chapter 6 . The above equation is somewhat similar to equation 2.78 in chapter 2 , although with a higher degree of accuracy. It draws a clearer picture of the effects influencing the perturbations. First we see again that the temperature perturbations observed today are not only affected by $\tilde{\Theta}_0(\eta_*)$ and $\tilde{\Theta}_1(\eta_*)$, but also $\tilde{\Psi}$. This is what we expect as photons lose energy by climbing out of gravity wells. This is of course another hint of the Sachs-Wolf effect, which we will calculate in section 6.4 . In addition, the last term quantifies the Integrated Sachs-Wolf effect, which describes the changes in the photon energy while passing through time-varying potentials.

4.12 Silk Dampening

There is one important aspect of the CMB anisotropies that we have not discussed yet, namely diffusion dampening [24] of the anisotropies. We will treat this with the same detail as we did for the acoustic oscillations in section 4.6, but we will present here the basic facts and ideas developed in the literature, extracting what we need to treat the CMB power spectrum in a more precise manner. Recall that in our tight-coupling approximation, we argued that the photons are tightly coupled to the baryon plasma through compton scattering of the electrons, which effectively reduced the mean free path of a photon to 0. This would be correct if the scattering rate was infinite, which of course it is not. Hence a fraction of the photons will free stream an amount before scattering. This will cause dampening of the acoustic oscillations, which is called *Silk dampening* (Silk 1968).

We can imagine the photons scattering of the electrons generating a path as a random walker. The mean free path of a generic photon being $\dot{\tau}^{-1} = (n_e\sigma_T)^{-1}$. During a Hubble time H^{-1} the photons scatter an amount $\dot{\tau} \cdot H^{-1}$. Thus the total distance λ_D traveled by a random walker is the mean free path times the square root of the scattering rate [22]. This leads us to $\lambda_D = (n_e\sigma_T)^{-1}\sqrt{n_e\sigma_T H^{-1}} = 1/\sqrt{n_e\sigma_T H}$. Hence a relevant dampening scale would be

$$k_D = \sqrt{n_e\sigma_T H}. \tag{4.130}$$

This means that the modes with wave number $k > k_D$ should be damped in some fashion. This is of course a very rough estimate which we will improve upon in the next section.

4.13 The Effect of Diffusion Dampening

There are two main diffusion effects [24] , viscous dampening from the quadropole $\tilde{\Theta}_2$ and heat conduction from the relative photon-baryon velocity $3\tilde{\Theta}_1/i - v_b$. Recall that we set the latter quantity to 0 at the end of section 4.11.2. We thus need to incorporate $\tilde{\Theta}_2$ in our work. We would now have to return to the Boltzmann Hierarchy (equation (4.25)) and include an equation for $\tilde{\Theta}_2$ in addition to the ones we had for $\tilde{\Theta}_0$ and $\tilde{\Theta}_1$ in the tight coupling approximation. Solving the resulting equations is a straight forward calculation (see [10]), where we basically obtain the following solutions

$$\tilde{\Theta}_0 \sim \exp\left(ik\int c_s d\eta\right)\exp\left(-\frac{k^2}{k_D^2}\right) \tag{4.131}$$

$$\tilde{\Theta}_1 \sim \exp\left(-ik\int c_s d\eta\right)\exp\left(-\frac{k^2}{k_D^2}\right). \tag{4.132}$$

As one can see, we recover the same solutions as in the tight-coupling limit, but with an additional exponential dampening factor, where there is basically no dampening for k values smaller than k_D. This was expected from the previous heuristic argument in section 4.12. However the dampening scale is now changed to

$$\frac{1}{k_D^2(\eta)} = \frac{1}{6}\int_0^\eta \frac{d\eta'}{(1+R)n_e\sigma_T a(\eta')}\left[\frac{R^2}{1+R} + \frac{8}{9}\right]. \tag{4.133}$$

An estimate for k_D is given in [10] where one takes the pre-recombination limit where are the electrons are free. The calculation leads to

$$k_D^{-2} \simeq 3.1{\cdot}10^6\mathrm{Mpc}^2 a^{5/2} f_D(a/a_{eq})(\Omega_b h^2)^{-1}\left(1-\frac{Y_p}{2}\right)^{-1}(\Omega_{m0}h^2)^{-1/2}. \tag{4.134}$$

Here Y_p is the mass fraction of helium with $Y_p \simeq 0.24$. f_D is a function we do not need as $f_d \to 1$ when a/a_{eq} becomes large. However this expression is not entirely correct at recombination, we will thus use a different approximation to k_D in our numerical work in section 7.2.

4.14 Finite Duration of Recombination

We end this chapter with a discussion of including the effects of diffusion dampening and the fact that recombination most probably was not an instantaneous process. The latter is often abbreviated by the finite thickness of the last scattering surface.

To include the diffusion effects we would in principle add a factor of $\exp\left(-\frac{k^2}{k_D^2}\right)$ in equation (4.125) and repeat our steps onwards. But because of the peak in the dampening factor about $\eta = \eta_*$, our delta function argument would no longer apply, leaving us with a complicated integral to solve. But thanks to Hu in [24] there is a way out of this dilemma. Recall that the visibility function $g(\eta)$ is basically a filter peaked about $\eta = \eta_*$, where the width of the peak is proportional to the thickness of the last scattering surface. In [24], Hu. et. al showed that including the finite thickness of the LSS would be accomplished by changing the dampening factor

$$D(k) = \int_0^{\eta_0} d\eta g(\eta) \exp\left(-\frac{k^2}{k_D^2(\eta)}\right). \tag{4.135}$$

Thus we multiply equation (4.125) by the above factor instead of $\exp\left(-\frac{k^2}{k_D^2}\right)$, now accounting for both effects.

Chapter 5

Growth of Inhomogeneities

5.1 Introduction

We will in this chapter investigate in a limited fashion some aspects of how the primordial inhomogeneities generated by inflation evolve to the large scale structure observed today. We will introduce some key aspects that has become convention in this field of study. But bear in mind that the treatment of this subject will be quite superficial, concentrating largely on concepts we require to increase our understanding of the CMB.

5.2 The Primordial Potential

As we saw in chapter 3, the quantum fluctuations in the Inflaton field gave rise to fluctuations in the potential $\Phi(\vec{k}, \eta)$. We will refer to the primordial value of the potential as $\Phi_p(\vec{k})$. Our aim is to see how this value relates to the potential Φ at later times. Observe that right after inflation ends, most modes k lie outside the horizon only to gradually re-enter the horizon as the universe expands. But as this happens the universe eventually goes from being radiation dominated to matter dominated after a_{eq}. Hence modes that re-enter the horizon prior to equality grow differently than the ones entering after. Once most modes are inside the horizon they all evolve equally, independent of wavelength.

We see that there are two distinct types of evolution, a wavelength dependent growth at “early” times, and a wavelength independent growth setting in at “late” times, which we refer to for the time being as a_{late}. It has thus become convention to separate these two growth patterns. The potential $\Phi(\vec{k}, a)$ can be expressed by $\Phi_p(\vec{k})$ by

$$\Phi(\vec{k},\eta) \propto \Phi_p(k) \times [\text{Growth function (k)}] \times [\text{Growth function (a)}]. \tag{5.1}$$

The wavelength dependent growth function is called *the Transfer Function*, denoted as $T(k)$. Following convention, it is defined in such a way that it is 1 at large scales. This means that

$$T(k) \equiv \frac{\Phi(k.a_{late})}{\Phi_{L-S}(k, a_{late})}\,, \tag{5.2}$$

where $\Phi_{L-S}(k, a_{late})$ is the large scale solution of the potential at a_{late}. We will in section 5.3 give explicit expressions for the transfer function .
Moving on to the wavelength independent growth at later times. It is defined as

$$D(a) \equiv a \cdot \frac{\Phi(a)}{\Phi(a_{late})}\,. \tag{5.3}$$

This function is simply known as *the Growth factor*. The growth factor *is* the matter perturbation δ, $D(a) \equiv \delta$, when most scales are well within the horizon, i.e. when $a > a_{late}$ (see [13, 10]). Our potential now looks like

$$\Phi(\vec{k},\eta) \propto \Phi_p(k) T(k) \frac{D(a)}{a}\,. \tag{5.4}$$

Concerning the proportionality factor, it is in fact 9/10, which follow from detailed calculations of the evolution equations for Φ on the large scale level. Indeed, it is the factor that the potential decreases as the universe becomes matter dominated. See [10] for details on the calculations. Hence we have

$$\Phi(\vec{k},\eta) = \frac{9}{10} \Phi_p(k) T(k) \frac{D(a)}{a}\,. \tag{5.5}$$

5.3 The Transfer Function

In the light of our recent discussions a natural question arises. When does the Transfer function regime end to leave $D(a)$ in charge of growth ? This brings k_{eq}, the mode with the wavenumber entering the horizon at equality, which we introduced in chapter 4, under a new light. It gives us a relevant scale at which $T(k)$ is dominant. Thus for $k \ll k_{eq} = a_{eq} H(a_{eq})$ we set $T(k) \sim 1$ and use $D(a)$ for studying the evolution of Φ. It has thus become natural to express the transfer function as a function of k/k_{eq}.
To calculate T(k) and the growth factor D(a), we would in principle have to

solve the full set of Boltzmann equation for the photons, matter and neutrinos coupled with the Einstein equations for Ψ and Φ. The equations in question, reproduced here for convenience, are

$$\dot{\Theta} + i\,k\mu\,\Theta + \dot{\Phi} + i\,k\mu\,\Psi = -\dot{\tau}\big[\Theta_0 - \Theta + \mu\,v_b\big] \tag{5.6}$$

$$\dot{v}_b + \frac{\dot{a}}{a}v_b = -ik\Psi + \frac{\dot{\tau}}{R}\left[3i\Theta_1 + v_b\right] \tag{5.7}$$

$$\dot{\delta}_b + ikv_b = -3\dot{\Phi} \tag{5.8}$$

$$\dot{v} + \mathcal{H}v = -ik\Psi \tag{5.9}$$

$$\dot{\delta} + ikv = -3\dot{\Phi} \tag{5.10}$$

$$k^2(\Phi + \Psi) = -32Ga^2\left[\rho_\gamma\Theta_2 + \rho_\nu\mathcal{N}_2\right] \tag{5.11}$$

$$k^2\Phi - 4\pi Ga^2\left[\rho_m\delta_m + 4\left(\rho_\gamma\Theta_0 + \rho_\nu\mathcal{N}_0\right)\right] = 12\pi G\frac{a^2H}{k}\left[i\rho_m v_m + 4\rho_r\Theta_r\right] \tag{5.12}$$

$$\dot{\mathcal{N}} + i\,k\mu\,\mathcal{N} = -\dot{\Phi} - i\,k\mu\,\Psi, \tag{5.13}$$

where $\rho_r = \rho_\gamma + \rho_\nu$ and $\Theta_r = \Theta_\gamma + \Theta_\nu$. There are many ways of attacking this problem, one way being to solve the full set numerically by ignoring the baryons as dark matter is dominant. One solution obtained by Bardeen et.al, known as the BBKS[1] transfer function[3] given by

$$T(x) = \frac{\ln(1 + 0.171x)}{0.171x}\left[1 + 0.284x + (1.18x)^2 + (0.399x)^3 + (0.490x)^4\right]^{-0.25} \tag{5.14}$$

where $x = k/k_{eq}$. But one can also obtain a semi-analytic solution by studying different extreme limits of the equations. Dodelsen does this in [10]. A small scale solution which we will use is

$$T(k) = 12\frac{k_{eq}^2}{k^2}\ln\left(\frac{k}{8k_{eq}}\right), \tag{5.15}$$

valid for $k \gg k_{eq}$. We will see this expression reappear in chapter 6 when studying the small scale CMB power spectrum.

5.4 The Growth Function

As mentioned earlier, after most interesting modes have entered the horizon, each wavelength k evolves in the same fashion [10]. A widely accepted time when this occurs corresponds to $z \sim 10$ which is approximately $a \sim 10^{-1}$.

[1] Bardeen, Bond, Kaiser and Szalay.

We will in this section find an expression for $D(a)$ valid for $a \gg a_{eq}$. At these times ρ_r is negligible compared to ρ_{dm}, and $\dot{\tau}$ is small. In addition as most modes are well within the horizon, we can take the small scale limit of the equations quantified by $\frac{a^2 H}{k} \ll 1$. The equations we are working with reduce to

$$\dot{v} + \frac{\dot{a}}{a} v = ik\Phi \tag{5.16}$$

$$\dot{\delta} + ikv = -3\dot{\Phi} \tag{5.17}$$

$$k^2\Phi = 4\pi G a^2 \rho_{dm} \delta. \tag{5.18}$$

We have also set $\Phi = -\Psi$ as anisotropic stress is small, and we are ignoring the baryons which is natural in this setting. We can set about to solve the above equations. We will do this by eliminating Φ and v from the picture to obtain one equation for δ. First we note that

$$\begin{aligned} 4\pi G \rho_{dm} \simeq 4\pi G \rho_m &= 4\pi G \rho_{cr} \Omega_{m0} a^{-3} \\ &= 4\pi G \frac{3H_0^2}{8\pi G} \Omega_{m0} a^{-3} \\ &= \frac{3}{2} H_0^2 \Omega_{m0} a^{-3}. \end{aligned} \tag{5.19}$$

This leads to

$$\dot{v} + \frac{\dot{a}}{a} v = ik\Phi \tag{5.20}$$

$$\dot{\delta} + ikv = -3\dot{\Phi} \tag{5.21}$$

$$k^2\Phi = \frac{3}{2} H_0^2 \Omega_{m0} a^{-1} \delta. \tag{5.22}$$

If we differentiate equation (5.21) we get

$$\ddot{\delta} + ik\dot{v} = -3\ddot{\Phi}. \tag{5.23}$$

Using equation (5.20) to eliminate $\dot{v}$ gives

$$\ddot{\delta} + ik\left(ik\Phi - \frac{\dot{a}}{a} v\right) = -3\ddot{\Phi}$$

$$\ddot{\delta} - k^2\Phi - \frac{\dot{a}}{a} ik\, v = -3\ddot{\Phi} \tag{5.24}$$

We can eliminate the velocity term by assuming that $v \gg \Phi$ in the small scales we are studying here. This leads us to conclude from equation (5.21) that $\dot{\delta} \simeq -ikv$, and implies that

$$\begin{aligned} \ddot{\delta} - k^2\Phi + \frac{\dot{a}}{a}\dot{\delta} &= -3\ddot{\Phi} \\ \ddot{\delta} + \frac{\dot{a}}{a}\dot{\delta} &= -3\ddot{\Phi} + k^2\Phi. \end{aligned} \tag{5.25}$$

Since $\ddot{\Phi}$ is of the order $\Phi/\eta^2 \sim \Phi/(aH)^2$, we can completely ignore it as $k^2\Phi$ is much larger in this small scale setting. Joined with equation (5.22) the above equation becomes

$$\ddot{\delta} + \frac{\dot{a}}{a}\dot{\delta} = \frac{3}{2}H_0^2\Omega_{m0}a^{-1}\delta. \tag{5.26}$$

We have successfully eliminated Φ and v and are left with a second order equation for δ. Observe that our equation involves some factors of a, hence a natural way we can simplify our problem is to change variables from η to a. To do this we need the differential operators

$$\frac{d}{d\eta} = \frac{da}{d\eta}\frac{d}{da} = \dot{a}\frac{d}{da} = a^2H\frac{d}{da} \tag{5.27}$$

$$\frac{d^2}{d\eta^2} = a^2H\frac{d}{da}\left(a^2H\frac{d}{da}\right) = a^2H\frac{d}{da}\left(a^2H\right)\frac{d}{da} + a^4H^2\frac{d^2}{da^2}. \tag{5.28}$$

Changing the variable to a turns equation (5.26) into

$$\begin{aligned} a^4H^2\delta'' + a^2H\frac{d}{da}\left(a^2H\right)\delta' + a^3H^2\delta' &= \frac{3}{2}H_0^2\Omega_{m0}a^{-1}\delta \\ \delta'' + \frac{1}{a^2H}\frac{d}{da}\left(a^2H\right)\delta' + a^{-1}\delta' &= \frac{3H_0^2\Omega_{m0}}{2H^2a^5}\delta \\ \delta'' + \left(\frac{1}{H}\frac{dH}{da} + 3a^{-1}\right)\delta' &= \frac{3H_0^2\Omega_{m0}}{2H^2a^5}\delta\,. \end{aligned} \tag{5.29}$$

Observe that primes indicate differentiation with respect to a. This new equation does not seem any simpler then equation (5.26), in fact quite the contrary. But this form will in fact allow us to find an analytical expression for $D(a)$. We start by guessing a solution of equation (5.29), then using this solution to find the interesting one, a standard approach in solving D.E.'s. The simplest guess being $\delta = H$.

5.4.1 Proof of the $D(a) = H$ Solution

We will now assume that $\delta = H$ is a solution of equation (5.29) and prove that this is indeed the case. Inserting this assumption in equation (5.29) gives

$$H'' + \left(\frac{1}{H}\frac{dH}{da} + 3a^{-1}\right)H' = \frac{3H_0^2\Omega_{m0}}{2Ha^5}. \tag{5.30}$$

At late times we can assume that the Hubble factor is of the form

$$H = H_0\sqrt{\Omega_{m0}a^{-3} + \Omega_\lambda}, \tag{5.31}$$

where we have in addition to a mass term a cosmological constant term. We are not excluding a possibility of dark energy by this assumption, as long as the change in ω_{de} is smaller than the change in a. Differentiating with respect to a gives

$$\frac{dH}{da} = -\frac{3}{2}H_0\frac{\Omega_{m0}a^{-4}}{\sqrt{\Omega_{m0}a^{-3} + \Omega_\lambda}} = -\frac{3}{2}\frac{H_0^2\Omega_{m0}}{a^4H} \tag{5.32}$$

$$\frac{d^2H}{da^2} = -\frac{3}{2}H_0^2\Omega_{m0}\left(-4\frac{a^{-5}}{H} - \frac{a^{-4}}{H^2}H'\right) = \frac{3}{2}\frac{H_0^2\Omega_{m0}}{Ha^5}\left(4 + \frac{aH'}{H}\right). \tag{5.33}$$

Inserting these results into equation (5.30) gives for the left hand side

$$\begin{aligned} LHS : &\frac{3}{2}\frac{H_0^2\Omega_{m0}}{Ha^5}\left(4 + \frac{aH'}{H}\right) - \left(\frac{H'}{H} + 3a^{-1}\right)\cdot\frac{3}{2}\frac{H_0^2\Omega_{m0}}{a^4H} \\ &= \frac{3}{2}\frac{H_0^2\Omega_{m0}}{Ha^5}\left[\left(4 + \frac{aH'}{H}\right) - \left(\frac{aH'}{H} + 3\right)\right] \\ &= \underline{\frac{3}{2}\frac{H_0^2\Omega_{m0}}{Ha^5}} \end{aligned} \tag{5.34}$$

The right hand side of equation (5.30) is indeed this factor, which proves that $\delta \equiv D(a) = H$ is a solution of equation (5.29). But is this the mode we are looking for ? A solution that grows as the Hubble factor is really not that interesting [10]. We will thus proceed in finding another solution.

5.4.2 Solving the Growth Equation

We will now find a different solution of equation (5.29) than the one found in the previous section. We start by introducing a new variable the $u = \delta/H$. Differentiation gives

$$\delta' = u'H + uH' \tag{5.35}$$

$$\delta'' = u''H + 2u'H' + uH''. \tag{5.36}$$

Inserting these into equation (5.29) leads to

$$
\begin{aligned}
u''H + 2u'H' + uH'' + \left(\frac{H'}{H} + 3a^{-1}\right)(u'H + uH') &= \frac{3H_0^2\Omega_{m0}}{2Ha^5}\, u \\
u''H + 3u'H' + 3a^{-1}u'H + \underbrace{\left(H'' + \left(\frac{H'}{H} + 3a^{-1}\right)H'\right)}_{=\frac{3H_0^2\Omega_{m0}}{2Ha^5}\text{ by eq.(5.30)}} u &= \frac{3H_0^2\Omega_{m0}}{2Ha^5}\, u \\
u''H + 3u'H' + 3a^{-1}u'H + \frac{3H_0^2\Omega_{m0}}{2Ha^5}\, u &= \frac{3H_0^2\Omega_{m0}}{2Ha^5}\, u,
\end{aligned}
\tag{5.37}
$$

which leads to

$$
\begin{aligned}
u''H + 3u'H' + 3a^{-1}u'H &= 0 \\
u'' + 3\left(\frac{H'}{H} + a^{-1}\right)u' &= 0.
\end{aligned}
\tag{5.38}
$$

We have obtained an equation that we can readily integrate. Integration gives

$$
\begin{aligned}
\ln u' &= -3\ln H - 3\ln a + const \\
u' &= \frac{C}{(aH)^3} \\
&\Downarrow \\
u &= C\int_0^a \frac{da'}{(a'H(a'))^3}
\end{aligned}
\tag{5.39}
$$

Recalling that $D(a) \equiv \delta$ at late times by construction, we get

$$
D(a) = CH\int_0^a \frac{da'}{(a'H(a'))^3} \tag{5.40}
$$

In (ref) it is shown that when matter is completely dominant , $D(a) = a$ is a solution of (5.29). This corresponds to a constant Φ. We can use this to find the integration constant C

$$
\begin{aligned}
D(a) &= C\,H\int_0^a \frac{da'}{(a'H(a'))^3} \\
&= C\,H\int_0^a \frac{da'}{(a'H_0\sqrt{\Omega_{m0}a'^{-3}}\,)^3} \\
&= C\,H\Omega_{m0}^{-3/2}H_0^{-3}\int_0^a a'^{\frac{3}{2}}\,da' \\
&= \frac{2}{5}\,C\,H\Omega_{m0}^{-3/2}H_0^{-3}a^{\frac{5}{2}}.
\end{aligned}
\tag{5.41}
$$

This is supposed to equal a. We get the equation

$$\begin{aligned}\frac{2}{5}C\,H\Omega_{m0}^{-3/2}H_0^{-3}a^{\frac{5}{2}} &= a\\ \Rightarrow C &= \frac{5}{2}\Omega_{m0}H_0^2\end{aligned} \tag{5.42}$$

We hence obtain the growth function

$$\begin{aligned}D(a) &= \frac{5}{2}\Omega_{m0}H_0^2 H\int_0^a \frac{da'}{(a'H(a'))^3}\\ &= \frac{5}{2}\Omega_{m0}\frac{H}{H_0}\int_0^a \frac{da'}{(a'H(a')/H_0)^3}\end{aligned} \tag{5.43}$$

We have now found a formula describing the clustering of matter at times after recombination. One can now readily calculate the growth function for different cosmological models. Of particular interest is of course the λCDM model with $\Omega_{m0} = 0.3$ and $\Omega_\lambda = 0.7$. We will calculate the growth factor for this cosmology in section 5.5, although we must do this numerically since in this case the above integral is a hypergeometric function not easily analyzed[2].

5.5 The Growth Function for some Cosmologies

In this section we will find the growth function for a flat λCDM model with $\Omega_{m0} = 0.3$ and $\Omega_\lambda = 0.7$. For a matter-dark energy model the Hubble factor is

$$H = H_0\sqrt{\Omega_{m0}a^{-3} + \Omega_{de}a^{-3(1+\omega)}}. \tag{5.44}$$

The growth function then takes the form

$$D(a) = \frac{5}{2}\Omega_{m0}\frac{H(a)}{H_0}\int_0^a r^{-3}\left(\Omega_{m0}r^{-3} + \Omega_{de}r^{-3(1+\omega)}\right)^{-\frac{3}{2}}dr \tag{5.45}$$

A cosmological constant universe has as we know $\omega = -1$. Unfortunately the above integral cannot be expressed in elementary functions, so we must turn to numerical techniques to solve this problem. It will also be instructive to compare our results with other universe models, like a completely matter dominated one and some dark energy models with different values of ω. Appendix B.1 shows a simple numerical integration code written in Python used to produce the different graphs shown below. Figure 5.1 shows the

[2]Curiously enough, the case with $\omega = -5/6$ *is* analytically solvable.

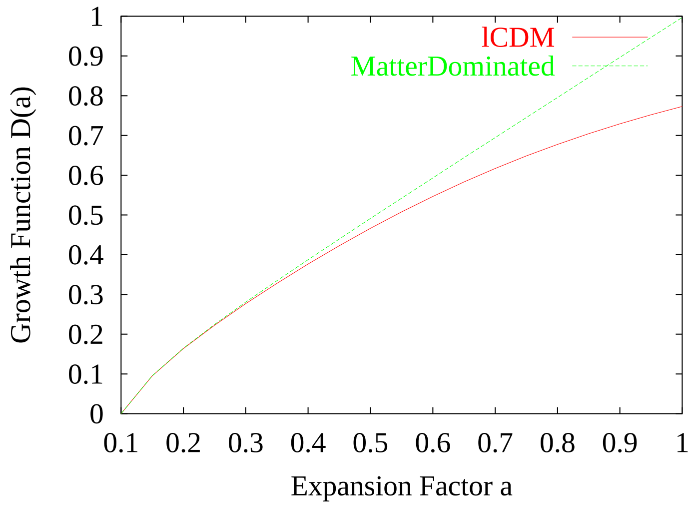

Figure 5.1: The growth factor for a ΛCDM model with $\Omega_{m0} = 0.3$ and $\Omega_\lambda = 0.7$, compared to a cosmology with $\Omega_{m0} = 1$.

growth function for a ΛCDM compared to a completely matter dominated flat universe with $\Omega_{m0} = 1$. We see here how at later times structure formation is damped a little by the presence of a cosmological constant.
We can also note in figure 5.2 the difference between a dark energy model with $\omega = -0.5$ compared to the other two models considered.

5.6 The Matter Power Spectrum

In the end of this chapter we will see what our recent development has implications to our previous work, specifically regarding the matter power spectrum we defined in chapter 3. How do we relate this to the primordial spectrum P_Φ generated by inflation. The formal definition [10] of P_Φ was

$$< \Phi_p(k)\Phi_p^*(k') >= (2\pi)^3\delta(\vec{k} - \vec{k}')P_\Phi \tag{5.46}$$

Recall that from chapter 3 we found that

$$P_\Phi = \frac{8\pi G}{9k^3}\frac{H^2}{\epsilon}\left(\frac{k}{aH}\right)^{n-1}. \tag{5.47}$$

We want to relate this to the matter power spectrum $< \delta(k)\delta^*(k') >= (2\pi)^3\delta(\vec{k} - \vec{k}')\mathcal{P}(k)$ at late times, i.e. $a > a_{late}$. To do this we start by

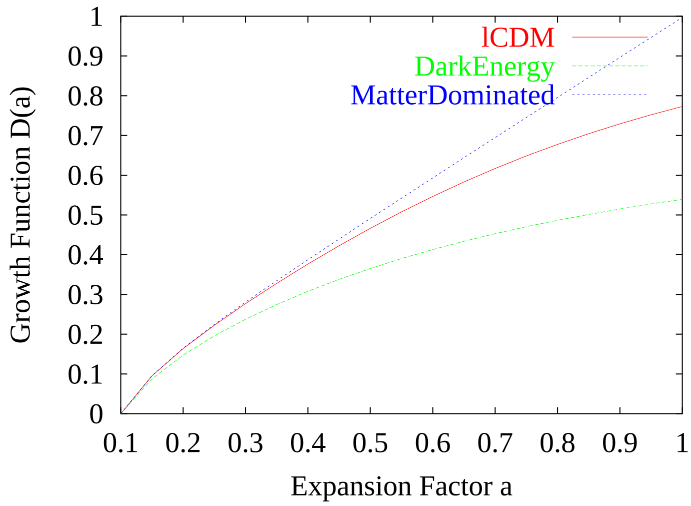

Figure 5.2: The growth factor for a ΛCDM model with $\Omega_{m0} = 0.3$ and $\Omega_\lambda = 0.7$, compared to a cosmology with $\Omega_{m0} = 1$ and a dark energy model with $\omega = -0.5$.

defining the *Scalar Amplitude*

$$\delta_H^2 \equiv \frac{8G}{\pi\epsilon}\left(\frac{H_0}{aH}\right)^{n-1}\left(\frac{HD(a_0)}{\frac{5}{2}\Omega_{m0}}\right)^2. \tag{5.48}$$

It has become convention to include the growth function into the definition of the scalar amplitude, the presence of H and Ω_{m0} renders the last factor dimensionless. Inserting this definition into equation (5.47) gives

$$P_\Phi = \frac{50\pi^2}{9k^3}\left(\frac{k}{H_0}\right)^{n-1}\delta_H^2\frac{\Omega_{m0}^2}{D^2(a_0)} \tag{5.49}$$

We need a relation between Φ and δ at late times. The natural choice is to use the sub-horizon equation we used before

$$k^2\Phi = 4\pi G\rho_m a^2\delta = \frac{3}{2}H_0^2\Omega_{m0}a^{-1}\delta, \tag{5.50}$$

which when solved for δ gives

$$\delta = \frac{2ak^2}{3H_0^2\Omega_{m0}}\Phi. \tag{5.51}$$

Using equation (5.5) we get

$$\begin{aligned}\delta &= \frac{2ak^2}{3H_0^2\Omega_{m0}} \cdot \frac{9}{10} T(k) \frac{D(a)}{a} \Phi_p \\ &= \frac{3k^2}{5H_0^2\Omega_{m0}} T(k) D(a) \Phi_p \end{aligned} \tag{5.52}$$

Taking the variance on both side's leads to

$$< \delta\delta^* > = \frac{9k^4}{25H_0^4\Omega_{m0}^2} T^2(k) D^2(a) < \Phi_p \Phi_p^* >, \tag{5.53}$$

which when combined with equation (5.49) gives

$$\begin{aligned}\mathcal{P}(k) &= \frac{9k^4}{25H_0^4\Omega_{m0}^2} T^2(k) D^2(a) \cdot \frac{50\pi^2}{9k^3} \left(\frac{k}{H_0} \right)^{n-1} \delta_H^2 \frac{\Omega_{m0}^2}{D^2(a_0)} \\ &\Downarrow \\ \mathcal{P}(k) &= 2\pi^2 \delta_H^2 \frac{k^n}{H_0^{n+3}} T^2(k) \left(\frac{D(a)}{D(a_0)} \right)^2 . \end{aligned} \tag{5.54}$$

This is the matter power spectrum in it's final form. We can see that the addition of $D(a)$ in the definition of δ_H gave the power spectrum a symmetric form. Although there are many ways in the literature to define the power spectrum and no clear standard have arisen at the time of this writing (see [8, 31, 16] for alternative ways to implement this). The physics is after all the same, so it boils down to taste and convenience. We have chosen the present approach as it is most suited for our work on the CMB power spectrum. Indeed as we will discover soon, equation (5.54) will play an integral part in chapter 6.

Chapter 6

The CMB Power Spectrum

6.1 Introduction

In this chapter we will investigate how our previous theoretical work can be turned into something measurable, namely the CMB power spectrum. We have referred to it numerous times throughout this work, it is thus time to deal with the concept in a quantitative manner.
Returning to first principles, we imagine that the temperature field of the universe $T(\bar{x}, \vec{n}, \eta)$ being uniform in the largest scales. As we did earlier, we expand the temperature into a zero order homogenous part and a first order part δT.

$$T(\bar{x}, \vec{n}, \eta) = T(\eta)\left[1 + \frac{\delta T(\bar{x}, \vec{n}, \eta)}{T}\right]. \tag{6.1}$$

The temperature field is mostly uniform, where the deviation from homogeneity is of the order $\frac{\delta T}{T} \equiv \Theta \sim 10^{-5}$. We can visualize the CMB-temperature field as defined on points on the spherical sky. It will thus be useful to expand Θ in spherical harmonics, being nothing else than functions defined on the sphere. This procedure will be analogous to a fourier expansion of the field.

6.2 The Power Spectrum

Although we have talked a great deal about it through this work, we have not really defined the CMB power spectrum. That is the aim of this section. We start by expanding the temperature contrast Θ in spherical harmonics[10].

$$\Theta(\bar{x}, \bar{n}, \eta) = \sum_{l=1}^{\infty}\sum_{m} a_{lm} Y_{lm}(\bar{n}) \tag{6.2}$$

This equation defines the spherical coefficients $a_{lm} \equiv a_{lm}(\bar{x}, \eta)$. These are in principle stochastic random variables originating from the quantum fluctuations set by inflation. We are also assuming that the fluctuations are Gaussian, like most inflation models predict. Hence the mean value $< a_{lm} >= 0$. One non-zero quantity we can extract[1] from the a_{lm}'s is the two-point correlation function, or simply the variance. This is given by

$$< a_{lm} a^*_{l'm'} >= \delta_{ll'} \delta_{mm'} C_l. \tag{6.3}$$

This equation defines the much talked about C_l, the *CMB Power Spectrum*. It is the sought after link between the theoretical and observational matters of the CMB.

6.2.1 Cosmic Variance

We must note however that there is a fundamental uncertainty when in measuring the C_l's which has been called *Cosmic variance*. The cosmic variance [12] is given by

$$\frac{\triangle C_l}{C_l} = \sqrt{\frac{2}{2l+1}}, \tag{6.4}$$

where $\triangle C_l$ is the variance of the difference between observed C_l and theoretical C_l. This means that for low l (large scales) our measurements will be inherently uncertain. This stems from the fact that we are observing only one sample of all the possible C_l. Having access to more universes would undoubtedly reduce this problem. But as we can see the cosmic variance reduces for higher l-values. Let us proceed in extracting an expression for C_l in terms of quantities acquired in previous chapters.

6.2.2 Spherical Expansion of a_{lm}

To find C_l we will need first to isolate a_{lm} from our given expressions and then square it [16]. Returning to equation (6.2), we can multiply it by $d\Omega Y^*_{lm}(\bar{n})$ and integrate to obtain

$$\begin{aligned} \int d\Omega Y^*_{lm}(\bar{n}) \Theta(\bar{x}, \bar{n}, \eta) &= \sum_{l'=1}^{\infty} \sum_{m'} a_{l'm'} \int d\Omega Y^*_{lm}(\bar{n}) Y_{l'm'}(\bar{n}) \\ &= \sum_{l'=1}^{\infty} \sum_{m'} a_{l'm'} \delta_{ll'} \delta_{mm'} \\ &= a_{lm}. \end{aligned} \tag{6.5}$$

[1]There are others, but beyond the scope of our work.

Recall now that our analysis of the Θ has mainly been done in Fourier space. We can imagine that we have an expression for Θ in Fourier space and want to relate it to it's value in real space, i.e. we need to invert the Fourier transformed value. Hence in our Fourier convention we get

$$a_{lm} = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int d\Omega Y^*_{lm}(\bar{n}) \tilde{\Theta}(\bar{k}, \bar{n}, \eta). \tag{6.6}$$

The complex conjugate is simply

$$a^*_{lm} = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \int d\Omega Y_{lm}(\bar{n}) \tilde{\Theta}^*(\bar{k}, \bar{n}, \eta). \tag{6.7}$$

To obtain an expression for C_l we must square a_{lm}. This gives

$$\begin{aligned} C_l &=< a_{lm} a^*_{lm} > \\ &= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int \frac{d^3k'}{(2\pi)^3} e^{-i\vec{k}'\cdot\vec{x}'} \\ &\quad \times \int d\Omega' Y^*_{lm}(\bar{n}') \int d\Omega Y_{lm}(\bar{n}) < \tilde{\Theta}(\bar{k}, \bar{n}) \tilde{\Theta}^*(\bar{k}', \bar{n}') > . \end{aligned} \tag{6.8}$$

The subtlety now is to incorporate the matter power spectrum from chapter 5 into our setting. We rewrite $\tilde{\Theta} = \delta \cdot \frac{\tilde{\Theta}}{\delta}$ where δ is the dark matter density contrast[10]. This allows us to write

$$\begin{aligned} < \tilde{\Theta}(\bar{k}, \bar{n}) \tilde{\Theta}^*(\bar{k}', \bar{n}') > &=< \delta(\vec{k}) \delta^*(\vec{k}') > \frac{\tilde{\Theta}(\bar{k})}{\delta(k)} \frac{\tilde{\Theta}^*(\bar{k}')}{\delta^*(k')} \\ &= (2\pi)^3 \delta(\vec{k} - \vec{k}') \mathcal{P}(k) \frac{\tilde{\Theta}(\bar{k})}{\delta(k)} \frac{\tilde{\Theta}^*(\bar{k}')}{\delta^*(k')}. \end{aligned} \tag{6.9}$$

Here we have used equation (5.54) from chapter 5 for the matter power spectrum. Inserting the above equation into equation (6.8) gives

$$\begin{aligned} C_l = &\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int d^3k' e^{-i\vec{k}'\cdot\vec{x}'} \delta(\vec{k} - \vec{k}') \mathcal{P}(k) \\ &\times \int d\Omega' Y^*_{lm}(\bar{n}') \int d\Omega Y_{lm}(\bar{n}) \frac{\tilde{\Theta}(\bar{k}, \bar{n})}{\delta(k)} \frac{\tilde{\Theta}^*(\bar{k}', \bar{n}')}{\delta^*(k')} \\ = &\int \frac{d^3k}{(2\pi)^3} \frac{\mathcal{P}(k)}{|\delta(k)|^2} \int d\Omega' Y^*_{lm}(\bar{n}') \int d\Omega Y_{lm}(\bar{n}) \tilde{\Theta}(\bar{k}, \bar{n}) \tilde{\Theta}^*(\bar{k}, \bar{n}'), \end{aligned} \tag{6.10}$$

where we have evaluated the k'-integral with the help of the delta function. Using the multipole expansion of $\tilde{\Theta}$ given by

$$\tilde{\Theta}(k, \bar{n}) = \sum_{l=0}^{\infty} (-i)^l (2l+1) P_l(\bar{k} \cdot \bar{n}) \tilde{\Theta}_l(k), \tag{6.11}$$

we obtain

$$
\begin{aligned}
C_l &= \int \frac{d^3k}{(2\pi)^3}\frac{\mathcal{P}(k)}{|\delta(k)|^2}\sum_{l'l''}(-i)^{l'}i^{l''}(2l'+1)(2l''+1)\tilde{\Theta}_{l'}\tilde{\Theta}^*_{l''} \\
&\quad \times \int d\Omega' Y^*_{lm}(\bar{n}')P_{l'}(\bar{k}\cdot\bar{n}') \times \int d\Omega Y_{lm}(\bar{n})P_{l''}(\bar{k}\cdot\bar{n}) \\
&\overbrace{=}^{eq.\,(C.20)} \int \frac{d^3k}{(2\pi)^3}\frac{\mathcal{P}(k)}{|\delta(k)|^2}\sum_{l'l''}(-i)^{l'}i^{l''}(2l'+1)(2l''+1)\tilde{\Theta}_{l'}\tilde{\Theta}^*_{l''} \\
&\quad \times Y^*_{lm}(\bar{k})Y_{lm}(\bar{k})\delta_{l'l}\delta_{l''l}\frac{(4\pi)^2}{(2l+1)^2} \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{\mathcal{P}(k)}{|\delta(k)|^2}|Y_{lm}(\bar{k})|^2(4\pi)^2|\tilde{\Theta}_l|^2. \qquad (6.12)
\end{aligned}
$$

Inserting $d^3k = dk\,k^2 d\Omega$ we finally get

$$
\begin{aligned}
C_l = C_l(\eta) &= \frac{2}{\pi}\int_0^\infty dk k^2\mathcal{P}(k)\Big|\frac{\tilde{\Theta}_l(k)}{\delta(k)}\Big|^2 \int d\Omega |Y_{lm}(\bar{k})|^2 \\
&= \frac{2}{\pi}\int_0^\infty dk k^2\mathcal{P}(k)\Big|\frac{\tilde{\Theta}_l(k)}{\delta(k)}\Big|^2, \qquad (6.13)
\end{aligned}
$$

since the last integral is 1 by orthogonality. Observe I have here written the explicit time dependance of C_l just to remind us that it is a time varying quantity.
We have now found an expression for the CMB power spectrum that can be used in numerical calculations. All one needs to do is to calculate the multipoles given by the Boltzmann hierarchy in chapter 4, coupled with a reasonable expression for the dark matter contrast $\delta(k)$. This is what is in principle done in CMBFast and other numerical codes, but they include much more of the more of the details like reionization and neutrinos, effects which I have not discussed here. What we will now do is to see how much information we can obtain from an analytic approach. Let us see how we can link the obtained expression with our previous work.

6.3 The C_l's

In the previous section we proved that

$$
C_l(\eta) = \frac{2}{\pi}\int_0^\infty dk k^2\mathcal{P}(k)\Big|\frac{\tilde{\Theta}_l(k)}{\delta(k)}\Big|^2, \qquad (6.14)
$$

where $\mathcal{P}(k)$ is the matter power spectrum and $\delta(k)$ is the matter density contrast. Recall that in chapter 5 we found an expression for the matter power spectrum which is

$$\mathcal{P}(k) = 2\pi^2 \delta_H^2 \frac{k^n}{H_0^{n+3}} T^2(k) \left(\frac{D(a)}{D(a_0)} \right)^2 , \tag{6.15}$$

where δ_H is the scalar amplitude set by inflation, $T(k)$ is the transfer function and $D(a)$ is the growth function , all of which where introduced in chapter 5 . We can insert this expression into equation (6.14) to obtain

$$\begin{aligned} C_l(\eta) &= \frac{2}{\pi} \int_0^\infty dk k^2 2\pi^2 \delta_H^2 \frac{k^n}{H_0^{n+3}} T^2(k) \left(\frac{D(a)}{D(a_0)} \right)^2 \Big| \frac{\tilde{\Theta}_l(k)}{\delta(k)} \Big|^2 \\ &= \frac{4\pi}{H_0^{n+3}} \delta_H^2 \left(\frac{D(a)}{D(a_0)} \right)^2 \int_0^\infty dk\, k^{2+n} T^2(k) \Big| \frac{\tilde{\Theta}_l(k)}{\delta(k)} \Big|^2 . \end{aligned} \tag{6.16}$$

We are measuring the power spectrum at $\eta = \eta_0$, we hence get

$$C_l \equiv C_l(\eta_0) = \frac{4\pi}{H_0^{n+3}} \delta_H^2 \int_0^\infty dk\, k^{2+n} T^2(k) \Big| \frac{\tilde{\Theta}_l(k, \eta_0)}{\delta(k, \eta_0)} \Big|^2 . \tag{6.17}$$

In the upcoming sections, we will see how we can find an expression for C_l, where the solution depends on what scales we are looking at.

6.4 Large Scale Solution : Sachs-Wolf Plataue

The large scale domain is characterized by small wavenumbers $k \ll 1$ which corresponds to large spatial dimensions. We must hence seek the small k limit of equation (6.17). The following calculation can also be viewed as a warm up exercise for the upcoming small scale calculation awaiting us in section 6.5. In the large scale limit [16], only the monopole term contributes in the free streaming solution we found in chapter 4 . This means that equation (4.129) reduces to

$$\tilde{\Theta}_l(\eta_0) = \left[\tilde{\Theta}_0(\eta_*) + \tilde{\Psi}(\eta_*) \right] j_l \left(k(\eta_0 - \eta_*) \right) . \tag{6.18}$$

Equation (6.17) thus becomes

$$C_l = \frac{4\pi}{H_0^{n+3}} \delta_H^2 \int_0^\infty dk\, k^{2+n} T^2(k) \frac{[\tilde{\Theta}_0(\eta_*) + \tilde{\Psi}(\eta_*)]^2}{|\, \delta(k, a_0)|^2} j_l^2 \left(k(\eta_0 - \eta_*) \right) . \tag{6.19}$$

We need to find an expression for the large scale monopole at the time of recombination. We can do this by going to the small k limit of equation (4.32) on page 89 which is simply

$$\begin{gathered}\dot{\tilde{\Theta}}_0 = -\dot{\tilde{\Phi}} \\ \Downarrow \\ \tilde{\Theta}(\eta) + \tilde{\Phi}(\eta) = C\,. \end{gathered} \tag{6.20}$$

Since we are considering adiabatic initial conditions, we have that $\tilde{\Theta}(0) = \tilde{\Phi}(0)/2$. This allows us to find the constant C above. We obtain

$$\begin{aligned}\tilde{\Theta}(0) + \tilde{\Phi}(0) &= C \\ \frac{\tilde{\Phi}(0)}{2} + \tilde{\Phi}(0) &= C \\ C &= \frac{3\tilde{\Phi}(0)}{2}\,, \end{aligned} \tag{6.21}$$

which allows us to write

$$\tilde{\Theta}_0(\eta) = -\tilde{\Phi}(\eta) + \frac{3\tilde{\Phi}(0)}{2}\,. \tag{6.22}$$

From [10] we obtain that the large scale solution of the gravitational potential is given by $\tilde{\Phi}(k,\eta_*) \simeq \frac{9\tilde{\Phi}(0)}{10}$. This means that

$$\begin{aligned}\tilde{\Theta}_0(k,\eta_*) &= -\tilde{\Phi}(k,\eta_*) + \frac{3\tilde{\Phi}(0)}{2} \\ &= -\tilde{\Phi}(k,\eta_*) + \frac{5\tilde{\Phi}(k,\eta_*)}{3} \\ &= \frac{2\tilde{\Phi}(k,\eta_*)}{3} \end{aligned} \tag{6.23}$$

We are interested in the quantity $\tilde{\Theta}_0(\eta_*)+\tilde{\Psi}(\eta_*)$. We are still ignoring setting anisotropic stress to 0, hence we get

$$\begin{aligned}\tilde{\Theta}_0(\eta_*) + \tilde{\Psi}(\eta_*) &= \frac{2\tilde{\Phi}(k,\eta_*)}{3} - \tilde{\Phi}(k,\eta_*) \\ &= -\frac{1}{3}\tilde{\Phi}(k,\eta_*) \end{aligned} \tag{6.24}$$

This equation is often referred to as the Sachs-Wolf effect. We have seen many different manifestations of this effect during our work, but we are finally in

a position to see what implications the Sachs-Wolf effect has on the CMB power spectrum.
Returning to equation (6.19), our new information leads us to

$$C_l = \frac{4\pi}{H_0^{n+3}}\,\delta_H^2 \int_0^\infty dk\,k^{2+n}T^2(k)\frac{\frac{1}{9}\tilde{\Phi}(\eta_*)^2}{|\,\delta(k,a_0)|^2} j_l^2\left(k(\eta_0-\eta_*)\right). \tag{6.25}$$

An additional simplification comes from that the transfer function $T(k) \equiv 1$ at large scales. We are thus left with

$$C_l = \frac{4\pi}{H_0^{n+3}}\,\delta_H^2 \int_0^\infty dk\,k^{2+n}\frac{\frac{1}{9}\tilde{\Phi}(\eta_*)^2}{|\,\delta(k,a_0)|^2} j_l^2\left(k(\eta_0-\eta_*)\right). \tag{6.26}$$

Observe now that the times which the gravitational potential $\tilde{\Phi}$ and the matter contrast δ are to be evaluated are different. One easy way of solving this is to use the growth function from chapter 5 to relate the potential today (a_0) with the potential at recombination (a_*). From equation (5.3) we get

$$\tilde{\Phi}(a_0) = \frac{\tilde{\Phi}(a_*)D(a_0)}{a_0}. \tag{6.27}$$

In addition, the matter perturbation $\delta(k,a_0)$ is related with the gravitational potential by

$$\delta(k,a_0) = \frac{k^2\Phi(a_0)a_0}{\frac{3}{2}\Omega_{m0}H_0^2}. \tag{6.28}$$

Taking these two equations together we obtain

$$\begin{aligned}
\frac{1}{3}\Phi(a_*) &= \frac{1}{3}\frac{\tilde{\Phi}(a_0)a_0}{D(a_0)} \\
&= \frac{1}{3}\frac{a_0}{D(a_0)}\cdot\frac{3}{2}\Omega_{m0}H_0^2\frac{\delta(k,a_0)}{k^2a_0} \\
&= \frac{1}{2}\frac{\Omega_{m0}H_0^2}{k^2D(a_0)}\,\delta(k,a_0).
\end{aligned} \tag{6.29}$$

Using the above result with equation (6.26) we get

$$\begin{aligned}
C_l &= \frac{4\pi}{H_0^{n+3}}\,\delta_H^2 \int_0^\infty dk\,k^{2+n}\frac{\frac{1}{4}\frac{\Omega_{m0}^2H_0^4}{k^4D^2(a_0)}\,|\,\delta(k,a_0)|^2}{|\,\delta(k,a_0)|^2} j_l^2\left(k(\eta_0-\eta_*)\right) \\
&= \frac{\pi\Omega_{m0}^2}{H_0^{n-1}D^2(a_0)}\,\delta_H^2 \int_0^\infty dk\,k^{n-2} j_l^2\left(k(\eta_0-\eta_*)\right)
\end{aligned} \tag{6.30}$$

Changing the variable to $x = k(\eta_0 - \eta_*)$ and using the fact that $\eta_0 \gg \eta_*$ we get

$$\begin{aligned} C_l &= \frac{\pi\Omega_{m0}^2}{H_0^{n-1}D^2(a_0)}\,\delta_H^2 \int_0^\infty dx \frac{x^{n-2}}{\eta_0^{n-1}} j_l^2(x) \\ &= \frac{\pi\Omega_{m0}^2}{(H_0\eta_0)^{n-1}D^2(a_0)}\,\delta_H^2 \int_0^\infty dx x^{n-2} j_l^2(x). \end{aligned} \tag{6.31}$$

We are left with an integral that is in fact analytically solvable . Most integral tables are tabulated for ordinary Bessel functions than the spherical bessel functions, we need a transform formula between the two. This is simply

$$j_l(x) = \sqrt{\frac{\pi/2}{x}} J_{l+\frac{1}{2}}(x), \tag{6.32}$$

where the $J_{l+\frac{1}{2}}(x)$ is an ordinary Bessel function. Using this result gives us

$$\begin{aligned} C_l &= \frac{\pi\Omega_{m0}^2\delta_H^2}{(H_0\eta_0)^{n-1}D^2(a_0)} \int_0^\infty dx x^{n-2} \frac{\pi/2}{x} J_{l+\frac{1}{2}}^2(x) \\ &= \frac{\pi^2\Omega_{m0}^2\delta_H^2}{2(H_0\eta_0)^{n-1}D^2(a_0)} \int_0^\infty dx x^{n-3} J_{l+\frac{1}{2}}^2(x) \end{aligned} \tag{6.33}$$

Referring to the tables given in section C.3, the above integral can be written in terms of gamma functions in the following manner

$$C_l^{LS} = \frac{\pi^2\Omega_{m0}^2\delta_H^2}{2(H_0\eta_0)^{n-1}D^2(a_0)} \frac{\Gamma(3-n)\Gamma(l+\frac{n}{2}-\frac{1}{2})}{2^{3-n}\Gamma^2(\frac{4-n}{2})\Gamma(l-\frac{n}{2}+\frac{5}{2})}. \tag{6.34}$$

We have finally obtained an expression for the large scale power spectrum, valid for low values of l. Equation (6.34) becomes particulary simple if the spectral index n is 1. This would lead to

$$\begin{aligned} C_l^{LS} &= \frac{\pi^2\Omega_{m0}^2\delta_H^2}{2D^2(a_0)} \frac{\Gamma(2)\Gamma(l+\frac{1}{2}-\frac{1}{2})}{2^2\Gamma^2(\frac{3}{2})\Gamma(l-\frac{1}{2}+\frac{5}{2})} \\ &= \frac{\pi^2\Omega_{m0}^2\delta_H^2}{2D^2(a_0)} \frac{\Gamma(l)}{4\cdot\frac{1}{4}\pi\Gamma(l+2)} \\ &= \frac{\pi\Omega_{m0}^2\delta_H^2}{2D^2(a_0)} \frac{(l-1)!}{(l+1)!} \\ &= \frac{\pi\Omega_{m0}^2\delta_H^2}{2D^2(a_0)} \frac{1}{l(l+1)}. \end{aligned} \tag{6.35}$$

It has become convention to display $l(l+1)C_l$ in the literature, mainly because if we multiply the above equation by $l(l+1)$ we get

$$l(l+1)C_l^{LS} = \frac{\pi\Omega_{m0}^2\delta_H^2}{2D^2(a_0)}. \tag{6.36}$$

Observe that this is a constant, hence the main contribution of the Sachs-Wolfe effect is to create a plateau in the CMB power spectrum[25]. The figures below shows $l(l+1)C_l^{LS}$ for a ΛCDM model. This concludes our work on the large scale power spectrum. The next section we will investigate the behavior of the small scale anisotropy which will be more computationally challenging than the large scale treatment we have done here.

6.5 Small Scale Solution

In smaller scales, both the monopole and the dipole have a significant effect on the spectrum. In chapter 4 we found the free streaming solution of the temperature perturbation $\tilde{\Theta}_l(\eta_0)$ given by

$$\begin{aligned}\tilde{\Theta}_l(\eta_0) = &\left[\tilde{\Theta}_0(\eta_*) + \tilde{\Psi}(\eta_*)\right] j_l\left(k(\eta_0-\eta_*)\right) \\ &+ 3\tilde{\Theta}_1(\eta_*)\Big[j_{l-1}\left(k(\eta_0-\eta_*)\right) - \frac{l+1}{k(\eta_0-\eta_*)} j_l\left(k(\eta_0-\eta)\right)\Big] \\ &+ \int_0^{\eta_0} d\eta\, e^{-\tau}\left(\dot{\tilde{\Psi}} - \dot{\tilde{\Phi}}\right) j_l\left(k(\eta_0-\eta)\right).\end{aligned} \tag{6.37}$$

We will use this result with equation (6.17) above, but we will at this stage simplify matters by omitting the integrated Sachs-Wolfe effect, i.e. ignoring the last integral. Setting $j_l\left(k(\eta_0-\eta)\right) \equiv j_l$ for simplicity, we get

$$\begin{aligned}\left|\tilde{\Theta}_l(\eta_0)\right|^2 &= \Big|\big[\tilde{\Theta}_0(\eta_*) + \tilde{\Psi}(\eta_*)\big]j_l + 3\tilde{\Theta}_1(\eta_*)\big[j_{l-1} - \frac{l+1}{k(\eta_0-\eta_*)}j_l\big]\Big|^2 \\ &= \big[\tilde{\Theta}_0(\eta_*) + \tilde{\Psi}(\eta_*)\big]^2 j_l^2 + 6\tilde{\Theta}_1(\eta_*)\big[\tilde{\Theta}_0(\eta_*) + \tilde{\Psi}(\eta_*)\big]\big[j_l j_{l-1} - \frac{l+1}{k(\eta_0-\eta_*)}j_l^2\big] \\ &\quad + 9\,\tilde{\Theta}_1(\eta_*)^2\big[j_{l-1}^2 - 2\frac{l+1}{k(\eta_0-\eta_*)}j_l j_{l-1} + \frac{(l+1)^2}{k^2(\eta_0-\eta_*)^2}j_l^2\big].\end{aligned} \tag{6.38}$$

We will from now onwards ease the notation by omitting the explicit time dependence in the perturbations. This will not create any confusion as long we remember that the perturbations are to be evaluated at recombination. Recall that also in chapter 4, we solved the Boltzmann equation for the monopole and dipole in the tight coupling approximation. These were given by

$$\tilde{\Theta}_0 + \tilde{\Psi} = (1+R)^{-\frac{1}{4}}\, \mathcal{C}_A \cos kr_s + R\tilde{\Phi} \tag{6.39}$$

$$\tilde{\Theta}_1 = \frac{(1+R)^{-\frac{3}{4}}}{\sqrt{3}} \mathcal{C}_A \sin kr_s. \tag{6.40}$$

Recall that at recombination, we can [24] approximate $R\tilde{\Phi}(\eta_*) \simeq R\tilde{\Phi}(0)\, T(k)$ where T(k) is the matter transfer function which we will write down in a moment. We will not include the effects of diffusion dampening at this time, leaving the treatment of the subject to section 6.6. In addition we shall at this time omit the cross term between the dipole and the monopole as this term will not contribute to our expression (see Appendix C.4 for justification of this). Squaring the above equations gives us

$$\begin{aligned} [\tilde{\Theta}_0 + \tilde{\Psi}]^2 = (1+R)^{-\frac{1}{2}}\, &\mathcal{C}_A^2 \cos^2 kr_s \\ &+ 2R\,(1+R)^{-\frac{1}{4}}\, \mathcal{C}_A \cos kr_s \tilde{\Phi}(0) T(k) \\ &+ R^2 \tilde{\Phi}(0)^2 T^2(k) \end{aligned} \tag{6.41}$$

$$\tilde{\Theta}_1^2 = \frac{(1+R)^{-\frac{3}{2}}}{3} \mathcal{C}_A^2 \sin^2 kr_s. \tag{6.42}$$

Reverting now to the initial conditions, in [42] it is shown that the small scale adiabatic initial condition is [2] $\mathcal{C}_A = 3\tilde{\Phi}(0)/2$. Inserting this into the above equations gives us

$$\begin{aligned} [\tilde{\Theta}_0 + \tilde{\Psi}]^2 = &\frac{9}{4}\tilde{\Phi}^2(0)\,(1+R)^{-\frac{1}{2}} \cos^2 kr_s \\ &+ 3\tilde{\Phi}^2(0) R\,(1+R)^{-\frac{1}{4}} \cos kr_s T(k) + \frac{9}{4}\tilde{\Phi}(0)^2 R^2 T^2(k) \end{aligned} \tag{6.43}$$

$$\tilde{\Theta}_1^2 = \frac{9}{4}\tilde{\Phi}^2(0) \frac{(1+R)^{-\frac{3}{2}}}{3} \sin^2 kr_s. \tag{6.44}$$

Inserting the above into equation (6.38) gives

$$\begin{aligned} \frac{4\left|\tilde{\Theta}_l(\eta_0)\right|^2}{9\,\tilde{\Phi}^2(0)} = &(1+R)^{-\frac{1}{2}} \cos^2(kr_s)\, j_l^2 + \frac{4}{3} R\,(1+R)^{-\frac{1}{4}} \cos kr_s T(k) j_l^2 + R^2 T^2(k) j_l^2 \\ &+ 3\,(1+R)^{-\frac{3}{2}} \sin^2(kr_s) \\ &\times \left[j_{l-1}^2 - 2\frac{l+1}{k(\eta_0-\eta_*)} j_l j_{l-1} + \frac{(l+1)^2}{k^2(\eta_0-\eta_*)^2} j_l^2 \right]. \end{aligned} \tag{6.45}$$

[2] Observe that this means we are setting up initial conditions in the radiation dominated era.

Returning now to the integral of equation (6.17), we see we need an expression for the dark matter density perturbation at η_*. We can do the same as we did in section 6.4 for the large scale case, i.e by relating the matter contrast at η_0 with the gravitational potential at recombination. Using equation (5.22) from chapter 5 and $\tilde{\Phi}(a_0) = \tilde{\Phi}(a_*)D(a_0)/a_0$ we get

$$\begin{aligned}\delta(k, a_0) &\simeq \frac{k^2}{\frac{3}{2}\Omega_{m0}H_0^2}\Phi(a_0)a_0 \\ &= \frac{k^2}{\frac{3}{2}\Omega_{m0}H_0^2}\tilde{\Phi}(a_*)D(a_0) \\ &\simeq= \frac{k^2}{\frac{3}{2}\Omega_{m0}H_0^2}\tilde{\Phi}(0)T(k)D(a_0)\end{aligned} \tag{6.46}$$

Inserting this into equation (6.17) implies that

$$\begin{aligned}C_l &= \frac{4\pi}{H_0^{n+3}}\delta_H^2 \int_0^\infty dk\, k^{2+n}T^2(k)\frac{\Theta_l^2(\eta_0, k)}{\frac{9}{4}\frac{k^4}{\Omega_{m0}^2 H_0^4}\Phi^2(0)T^2(k)D^2(a_0)} \\ &= \frac{4\pi}{H_0^{n-1}}\frac{\Omega_{m0}^2}{D^2(a_0)}\delta_H^2 \int_0^\infty dk\, k^{n-2}\frac{4\Theta_l^2(\eta_0, k)}{9\Phi^2(0)}.\end{aligned} \tag{6.47}$$

We can make further simplifications to the above integral. Since $\eta_0 \gg \eta_*$, we can set the Bessel function argument to $k\eta_0$. This leads us to introduce the variable $x = k\eta_0$ into the integral. This substitution gives

$$\begin{aligned}C_l = \frac{4\pi}{(\eta_0 H_0)^{n-1}}\frac{\Omega_{m0}^2}{D^2(a_0)}\delta_H^2 \int_0^\infty dx\, x^{n-2} \times \Bigg[&(1+R)^{-\frac{1}{2}}\cos^2(xr_s/\eta_0)\, j_l^2 \\ &+ \frac{4}{3}R\,(1+R)^{-\frac{1}{4}}\cos(xr_s/\eta_0)T(x/\eta_0)j_l^2 \\ &+ R^2T^2(x/\eta_0)j_l^2 + 3\,(1+R)^{-\frac{3}{2}}\sin^2(xr_s/\eta_0) \\ &\times \left[j_{l-1}^2 - 2\frac{l+1}{x}j_l j_{l-1} + \frac{(l+1)^2}{x^2}j_l^2\right]\Bigg],\end{aligned} \tag{6.48}$$

where we have inserted the expressions obtained for the monopole and dipole, and we have as mentioned earlier omitted the dipole-monopole cross term. To be able to calculate the above integrals, we will first need to relate the spherical Bessel functions to ordinary Bessel functions as most integral tables give only formulas for the latter. The relation between a spherical function $j_l(x)$ and an ordinary Bessel function $J_\nu(x)$ is given by

$$j_l(x) = \sqrt{\frac{\pi/2}{x}}J_{l+\frac{1}{2}}(x) \tag{6.49}$$

We can easily see that using this relation with equation (6.48) will infer an extra factor of $\pi/(2x)$ throughout the entire integrand as each term includes products of two Bessel functions. We thus obtain

$$\begin{aligned} C_l = \frac{2\pi^2}{(\eta_0 H_0)^{n-1}} \frac{\Omega_{m0}^2}{D^2(a_0)} \delta_H^2 \int_0^\infty dx\, x^{n-3} \times \Bigg[& (1+R)^{-\frac{1}{2}} \cos^2(xr_s/\eta_0)\, J_{l+1/2}^2 \\ & + \frac{4}{3} R\,(1+R)^{-\frac{1}{4}} \cos(xr_s/\eta_0) T(x/\eta_0) J_{l+1/2}^2 \\ & + R^2 T^2(x/\eta_0) J_{l+1/2}^2 + 3\,(1+R)^{-\frac{3}{2}} \sin^2(xr_s/\eta_0) \\ & \times \left[J_{l-1/2}^2 - 2\frac{l+1}{x} J_{l+1/2} J_{l-1/2} + \frac{(l+1)^2}{x^2} J_{l+1/2}^2 \right] \Bigg]. \end{aligned} \tag{6.50}$$

Before we move on to calculate the integral we can make one additional simplification. Recall that the angular scale l corresponds roughly to $x = k\eta_0$ in our flat cosmology. At small scales (large x), the Bessel function $J^2_{l+\frac{1}{2}}(x)$ fluctuates much faster than the sine and cosine functions. Hence we can, at least as a first approximation take $\sin^2(xr_s/\eta_0)$ and $\cos^2(xr_s/\eta_0)$ to be constant with the identification $x \to l$. We can therefore set

$$\int_0^\infty dx\, x^{n-3} \cos^2(xr_s/\eta_0) J^2_{l+\frac{1}{2}} \simeq \cos^2(lr_s/\eta_0) \int_0^\infty dx\, x^{n-3} J^2_{l+\frac{1}{2}}, \tag{6.51}$$

in our integral, with equivalent approximations for all the occurring trigonometric functions[3]. Using this simplification leads to

$$\begin{aligned} C_l \simeq \frac{2\pi^2}{(\eta_0 H_0)^{n-1}} \frac{\Omega_{m0}^2}{D^2(a_0)} \delta_H^2 \int_0^\infty dx\, x^{n-3} \times \Bigg[& (1+R)^{-\frac{1}{2}} \cos^2(lr_s/\eta_0)\, J_{l+1/2}^2 \\ & + \frac{4}{3} R\,(1+R)^{-\frac{1}{4}} \cos(lr_s/\eta_0) T(x/\eta_0) J_{l+1/2}^2 \\ & + R^2 T^2(x/\eta_0) J_{l+1/2}^2 + 3\,(1+R)^{-\frac{3}{2}} \sin^2(lr_s/\eta_0) \\ & \times \left[J_{l-1/2}^2 - 2\frac{l+1}{x} J_{l+1/2} J_{l-1/2} + \frac{(l+1)^2}{x^2} J_{l+1/2}^2 \right] \Bigg]. \end{aligned} \tag{6.52}$$

For convenience we will define

$$Y^{-1} \equiv \frac{2\pi^2}{(\eta_0 H_0)^{n-1}} \frac{\Omega_{m0}^2}{D^2(a_0)} \delta_H^2, \tag{6.53}$$

[3]We will make a similar approximation for the logarithmic terms appearing in the transfer function.

just to ease the notation. In addition ,we will for the spectral index n make the natural choice of $n \equiv 1$, corresponding to the Harrison-Zeldovich spectrum. We are thus left with

$$\begin{aligned} C_l Y \simeq \int_0^\infty dx\, x^{-2} \times \Bigg[& (1+R)^{-\frac{1}{2}} \cos^2(lr_s/\eta_0)\, J^2_{l+1/2} \\ & + \frac{4}{3} R\,(1+R)^{-\frac{1}{4}} \cos(lr_s/\eta_0) T(x/\eta_0) J^2_{l+1/2} \\ & + R^2 T^2(x/\eta_0) J^2_{l+1/2} + 3\,(1+R)^{-\frac{3}{2}} \sin^2(lr_s/\eta_0) \\ & \times \left[J^2_{l-1/2} - 2\frac{l+1}{x} J_{l+1/2} J_{l-1/2} + \frac{(l+1)^2}{x^2} J^2_{l+1/2} \right] \Bigg]. \end{aligned} \tag{6.54}$$

6.5.1 Matter Transfer Function

As we can see from equation (6.54), the expression for the C_l's include the matter transfer function $T(k)$. We need a transfer function that includes the effects of Baryons as well as dark matter. A reasonable way to do this is to split up the matter transfer function into two distinct parts

$$T(k) = \left(1 - \frac{\Omega_b}{\Omega_{m0}}\right) T_c(k) + \frac{\Omega_b}{\Omega_{m0}} T_b(k), \tag{6.55}$$

where $T_c(k)$ is the (cold) dark matter transfer function and $T_b(k)$ is the baryonic transfer function. We have already introduced the small scale dark matter transfer function in chapter 5 which is given by

$$T_c(k) \simeq 12 \frac{k_{eq}^2}{k^2} \ln\left(\frac{k}{8k_{eq}}\right) \tag{6.56}$$

For the baryonic transfer function we turn to the expression obtained by Eisenstein et al in [13]. This is

$$T_b(k) \simeq 2.07\,(1+R)^{-\frac{3}{4}} \frac{k_{eq}}{k} \sin(kr_s) G\, D_l. \tag{6.57}$$

Here, D_l is the diffusion dampening term which we will obtain later in section 6.6. The term G in our notation is given by

$$G = \frac{a_*}{a_{eq}} \left[-6\sqrt{1 + \frac{a_*}{a_{eq}}} + \left(2 + 3\frac{a_*}{a_{eq}}\right) \ln\left(\frac{\sqrt{1+\frac{a_*}{a_{eq}}} + 1}{\sqrt{1+\frac{a_*}{a_{eq}}} - 1} \right) \right]. \tag{6.58}$$

The appearance of the 'sine' term is a direct manifestation of the so called *velocity overshoot* effect. As recombination occurs, the baryons are released from the photon-coupling and move kinematically according to their velocity. This generates additional perturbations which are proportional to their velocity $v_b \sim \tilde{\Theta}_1$.

We shall now use these expressions with equation (6.54). To ease the notation we will set $\sigma \equiv 1 - \frac{\Omega_b}{\Omega_{m0}}$ and $\nu \equiv \frac{\Omega_b}{\Omega_{m0}}$ and define $l_{eq} \equiv k_{eq}\eta_0$. We will thus need the following expressions

$$T(x/\eta_0) = 12\sigma \frac{l_{eq}^2}{x^2} \ln\left(\frac{x}{8l_{eq}}\right) + 2.07\nu\,(1+R)^{-\frac{3}{4}} \frac{l_{eq}}{x} \sin(xr_s/\eta_0) G\, D_l \quad (6.59)$$

$$\begin{aligned} T^2(x/\eta_0) = {} & 144\sigma^2 \frac{l_{eq}^4}{x^4} \ln^2\left(\frac{x}{8l_{eq}}\right) \\ & + 49.68\,(1+R)^{-\frac{3}{4}} \sigma\nu \frac{l_{eq}^3}{x^3} \ln\left(\frac{x}{8l_{eq}}\right) \sin(xr_s/\eta_0) G\, D_l \\ & + 4,2849\nu^2\,(1+R)^{-\frac{3}{2}} \frac{l_{eq}^2}{x^2} \sin^2(xr_s/\eta_0) G^2 D_l^2 \end{aligned} \quad (6.60)$$

Inserting these expressions into equation (6.54) gives

$$\begin{aligned} C_l Y \simeq {} & (1+R)^{-\frac{1}{2}} \cos^2(lr_s/\eta_0) \int_0^\infty dx\, x^{-2} J_{l+1/2}^2 \\ & + 12\frac{4}{3} R\,(1+R)^{-\frac{1}{4}}\, \sigma \cos(lr_s/\eta_0) \int_0^\infty dx\, J_{l+1/2}^2 \frac{l_{eq}^2}{x^4} \ln\left(\frac{x}{8l_{eq}}\right) \\ & + 2.07\frac{4}{3} R\,(1+R)^{-1} \cos(lr_s/\eta_0)\nu \int_0^\infty dx J_{l+1/2}^2 \frac{l_{eq}}{x^3} \sin(xr_s/\eta_0) G\, D_l \\ & + 144\sigma^2 R^2 \int_0^\infty dx J_{l+1/2}^2 \frac{l_{eq}^4}{x^6} \ln^2\left(\frac{x}{8l_{eq}}\right) \\ & + 49.68 R^2\,(1+R)^{-\frac{3}{4}} \sigma\nu \int_0^\infty dx J_{l+1/2}^2 \frac{l_{eq}^3}{x^5} \ln\left(\frac{x}{8l_{eq}}\right) \sin(xr_s/\eta_0) G\, D_l \\ & + 4,2849\nu^2 R^2\,(1+R)^{-\frac{3}{2}} \int_0^\infty dx J_{l+1/2}^2 \frac{l_{eq}^2}{x^4} \sin^2(xr_s/\eta_0) G^2 D_l^2 \\ & + 3\,(1+R)^{-\frac{3}{2}} \sin^2(lr_s/\eta_0) \int_0^\infty dx x^{-2} \big[J_{l-1/2}^2 - 2\frac{l+1}{x} J_{l+1/2} J_{l-1/2} \\ & \qquad + \frac{(l+1)^2}{x^2} J_{l+1/2}^2\big]. \end{aligned} \quad (6.61)$$

We can again simplify our problem bit further by noting that for large x, $\ln x$ and $\ln^2 x$ are approximately constant (as $d \ln^m x/dx \to 0$ for large x). Since the Bessel functions oscillate much more than the logarithm, we will set

$$\int_0^\infty dx\, x^{-\lambda} \ln(x) J^2_{l+\frac{1}{2}} \simeq \ln l \int_0^\infty dx\, x^{-\lambda} J^2_{l+\frac{1}{2}}, \tag{6.62}$$

as we did for the trigonometric function case. Using this approximation we obtain

$$\begin{aligned}
C_l Y \simeq\; & (1+R)^{-\frac{1}{2}} \cos^2(lr_s/\eta_0) \int_0^\infty dx\, x^{-2} J^2_{l+1/2} \\
& + 12\frac{4}{3} R\,(1+R)^{-\frac{1}{4}}\, \sigma \cos(lr_s/\eta_0) \ln\left(\frac{l}{8l_{eq}}\right) \int_0^\infty dx\, J^2_{l+1/2} \frac{l^2_{eq}}{x^4} \\
& + 2.07\frac{4}{3} R\,(1+R)^{-1} \cos(lr_s/\eta_0) \nu \sin(lr_s/\eta_0) G\, D_l \int_0^\infty dx J^2_{l+1/2} \frac{l_{eq}}{x^3} \\
& + 144\sigma^2 R^2 \ln^2\left(\frac{l}{8l_{eq}}\right) \int_0^\infty dx J^2_{l+1/2} \frac{l^4_{eq}}{x^6} \\
& + 49.68 R^2\,(1+R)^{-\frac{3}{4}}\, \sigma\nu \ln\left(\frac{l}{8l_{eq}}\right) \sin(lr_s/\eta_0) G\, D_l \int_0^\infty dx J^2_{l+1/2} \frac{l^3_{eq}}{x^5} \\
& + 4,2849\nu^2 R^2\,(1+R)^{-\frac{3}{2}} \sin^2(lr_s/\eta_0) G^2 D_l^2 \int_0^\infty dx J^2_{l+1/2} \frac{l^2_{eq}}{x^4} \\
& + 3\,(1+R)^{-\frac{3}{2}} \sin^2(lr_s/\eta_0) \int_0^\infty dx x^{-2} \big[J^2_{l-1/2} - 2\frac{l+1}{x} J_{l+1/2} J_{l-1/2} \\
& \qquad\qquad + \frac{(l+1)^2}{x^2} J^2_{l+1/2} \big].
\end{aligned} \tag{6.63}$$

We are now in a position to be able to find an expression for C_l. But first we will need to know some Bessel integrals.

6.5.2 Calculation of C_l

We have now all the tools we need to calculate C_l. All the remaining integrals in equation (6.63) correspond exactly to the ones given in section C.3. By refereing to these equations we can easily write down the values of the integrals

$$
\begin{aligned}
C_l Y \simeq\ & (1+R)^{-\frac{1}{2}} \cos^2(lr_s/\eta_0) \frac{1!(l-1)!}{2^2\Gamma^2(3/2)(l+1)!} \\
& + 12\frac{4}{3} R\,(1+R)^{-\frac{1}{4}}\, \sigma \cos(lr_s/\eta_0) l_{eq}^2 \ln\left(\frac{l}{8l_{eq}}\right) \frac{3!(l-2)!}{2^4\Gamma^2(5/2)(l+2)!} \\
& + 2.07\frac{4}{3} R\,(1+R)^{-1} \cos(lr_s/\eta_0)\nu \sin(lr_s/\eta_0) G\, D_l l_{eq} \frac{2!\Gamma(l-1/2)}{2^3 1!1!\Gamma(l+5/2)} \\
& + 144\sigma^2 R^2 l_{eq}^4 \ln^2\left(\frac{l}{8l_{eq}}\right) \frac{5!(l-3)!}{2^6\Gamma^2(7/2)(l+3)!} \\
& + 49.68R^2\,(1+R)^{-\frac{3}{4}}\, \sigma\nu l_{eq}^3 \ln\left(\frac{l}{8l_{eq}}\right) \sin(lr_s/\eta_0) G\, D_l \frac{4!\Gamma(l-3/2)}{2^5 2!2!\Gamma(l+5/2)} \\
& + 4,2849\nu^2 R^2\,(1+R)^{-\frac{3}{2}} \sin^2(lr_s/\eta_0) G^2 l_{eq}^2 D_l^2 \frac{3!(l-2)!}{2^4\Gamma^2(5/2)(l+2)!} \\
& + 3\,(1+R)^{-\frac{3}{2}} \sin^2(lr_s/\eta_0) \\
& \qquad \times \Big[\frac{1!(l-2)!}{2^2\Gamma^2(3/2)l!} - \frac{2(l+1)2!(l-2)!}{2^3\Gamma(5/2)\Gamma(3/2)(l+1)!} \\
& \qquad\qquad + \frac{(l+1)^2 3!(l-2)!}{2^4\Gamma^2(5/2)(l+2)!}\Big].
\end{aligned} \tag{6.64}
$$

We are as in the large scale case interested in displaying $l(l+1)C_l$ instead of simply C_l. We can hence sort out all the factorials by using the Gamma function relations $\Gamma(x+1) = x\Gamma(x)$ with $\Gamma(1/2) = \sqrt{\pi}$. In addition we multiply equation (6.64) by $\pi l(l+1)$ and go the large l limit. We are left with

$$
\begin{aligned}
\pi l(l+1) C_l Y \simeq\ & (1+R)^{-\frac{1}{2}} \cos^2(lr_s/\eta_0) \\
& + \frac{32}{3} R\,(1+R)^{-\frac{1}{4}}\, \sigma \cos(lr_s/\eta_0) \frac{l_{eq}^2}{l^2} \ln\left(\frac{l}{8l_{eq}}\right) \\
& + 0.69\pi R\,(1+R)^{-1} \cos(lr_s/\eta_0)\nu \sin(lr_s/\eta_0) G\, D_l \frac{l_{eq}}{l} \\
& + 76.8\sigma^2 R^2 \frac{l_{eq}^4}{l^4} \ln^2\left(\frac{l}{8l_{eq}}\right) \\
& + 9.315\pi R^2\,(1+R)^{-\frac{3}{4}}\, \sigma\nu \frac{l_{eq}^3}{l^3} \ln\left(\frac{l}{8l_{eq}}\right) \sin(lr_s/\eta_0) G\, D_l \\
& + 2.8556\nu^2 R^2\,(1+R)^{-\frac{3}{2}} \sin^2(lr_s/\eta_0) G^2 \frac{l_{eq}^2}{l^2} D_l^2 \\
& + (1+R)^{-\frac{3}{2}} \sin^2(lr_s/\eta_0).
\end{aligned} \tag{6.65}
$$

We have almost reached our goal. The only thing remaining to account for Silk dampening.

6.6 Small Scale Diffusion Dampening

Recall that in chapter 4 we quantified the effect of diffusion dampening and the finite thickness of the LSS by the inclusion of the function

$$D(k) = \int_0^{\eta_0} d\eta \, g(\eta) \exp\left(-\frac{k^2}{k_D^2(\eta)}\right). \tag{6.66}$$

Hence we would have to include this factor in the calculation of the C_l's earlier. The natural thought one might have is to make the projection to l-space by setting $x \to l$ as we did earlier. It turns out from [24] that the dampening effect can be approximated in l-space by

$$D_l = \exp\left[-(l/l_D)^{-m}\right], \tag{6.67}$$

where $l_D = k_D\eta_0$ and

$$m = a_3(\Omega_b h^2)^{a_4}\left[1 + (\Omega_b h^2)^{1.8}\right]^{1/5} \tag{6.68}$$
$$a_3 = 1.03(\Omega_{m0}h^2)^{0.0335} \tag{6.69}$$
$$a_4 = -0.0473(\Omega_{m0}h^2)^{-0.0639}, \tag{6.70}$$

which are parameters obtained in a best-fit power law estimation. Plugging in the numbers for a ΛCDM model we obtain $m \simeq 1.2$ which is slightly less than the expected $m = 2$ by a $k\eta_0 \to l$ projection. This is due to the finite thickness effect we included by virtue of the visibility function that serves to smooth out the dampening. Thus we only need add a factor of D_l in our acquired expression for the C_l's . The final expression we obtain is thus

$$\begin{aligned}
\pi l(l+1)C_l Y \simeq\ & (1+R)^{-\frac{1}{2}}\cos^2(lr_s/\eta_0)D_l^2 \\
& + \frac{32}{3}\sigma R\,(1+R)^{-\frac{1}{4}}\cos(lr_s/\eta_0)\frac{l_{eq}^2}{l^2}\ln\left(\frac{l}{8l_{eq}}\right)D_l \\
& + 0.69\pi\nu R\,(1+R)^{-1}\cos(lr_s/\eta_0)\sin(lr_s/\eta_0)G\,D_l^2\frac{l_{eq}}{l} \\
& + 76.8\sigma^2 R^2\frac{l_{eq}^4}{l^4}\ln^2\left(\frac{l}{8l_{eq}}\right) \\
& + 9.315\pi\nu\sigma R^2\,(1+R)^{-\frac{3}{4}}\frac{l_{eq}^3}{l^3}\ln\left(\frac{l}{8l_{eq}}\right)\sin(lr_s/\eta_0)G\,D_l \\
& + 2.8556\nu^2 R^2\,(1+R)^{-\frac{3}{2}}\sin^2(lr_s/\eta_0)G^2\frac{l_{eq}^2}{l^2}D_l^2 \\
& + (1+R)^{-\frac{3}{2}}\sin^2(lr_s/\eta_0)D_l^2.
\end{aligned} \tag{6.71}$$

We have finally reached our goal. This is our approximation for the CMB power spectrum at small scales. In chapter 7 we will outline how to simulate the graph of C_l and see what we can learn from it. Recall that all the functions on the right hand side are to be evaluated at $\eta = \eta_*$.

Chapter 7

Discussion and Conclusions

7.1 Introduction

We will in this chapter study the implications of our previous work, focusing mainly on the small scale CMB power spectrum obtained in chapter 6. Although our somewhat limited model for the C_l's have been approximate at the least, we will see that our results are mostly in accordance with the power spectrum obtained from numerical codes such as CMBFast or CMBEasy. Recall that we have omitted the Integrated Sachs-Wolfe effect (late and early), neutrinos and Reionization into our approximation. We will mention some of the effects that occur with the power spectrum towards the end of this chapter.

7.2 Numerical Assumptions and Relations

We will in this section outline the numerics necessary to plot the small scale C_l's. To clarify what we need we can insert the explicit time dependencies back into equation (6.71) from chapter 6. This gives

$$\begin{aligned}\pi l(l+1)C_l Y \simeq\ & (1+R(\eta_*))^{-\frac{1}{2}}\cos^2(lr_s(\eta_*)/\eta_0)D_l^2 \\ &+ \frac{32}{3}\sigma R(\eta_*)\,(1+R(\eta_*))^{-\frac{1}{4}}\cos(lr_s(\eta_*)/\eta_0)\frac{l_{eq}^2}{l^2}\ln\left(\frac{l}{8l_{eq}}\right)D_l \\ &+ 0.69\pi\nu R\,(1+R(\eta_*))^{-1}\cos(lr_s(\eta_*)/\eta_0)\sin(lr_s(\eta_*)/\eta_0)G\,D_l^2\frac{l_{eq}}{l} \\ &+ 76.8\sigma^2 R^2(\eta_*)\frac{l_{eq}^4}{l^4}\ln^2\left(\frac{l}{8l_{eq}}\right) \\ &+ 9.315\pi\nu\sigma R^2(\eta_*)\,(1+R(\eta_*))^{-\frac{3}{4}}\frac{l_{eq}^3}{l^3}\ln\left(\frac{l}{8l_{eq}}\right)\sin(lr_s(\eta_*)/\eta_0)G\,D_l \\ &+ 2.8556\nu^2 R^2(\eta_*)\,(1+R(\eta_*))^{-\frac{3}{2}}\sin^2(lr_s(\eta_*)/\eta_0)G^2\frac{l_{eq}^2}{l^2}D_l^2 \\ &+ (1+R(\eta_*))^{-\frac{3}{2}}\sin^2(lr_s(\eta_*)/\eta_0)D_l^2,\end{aligned} \tag{7.1}$$

where Y was defined as

$$Y^{-1} \equiv 2\pi^2\frac{\Omega_{m0}^2}{D^2(a_0)}\delta_H^2. \tag{7.2}$$

We need to calculate the baryon-photon ratio R, the sound horizon $r_s(\eta)$ and the Growth function $D(a)$, where most of them are to be evaluated at $\eta = \eta_*$. Firstly for the epoch of recombination we will use the expression obtained for the conformal time in chapter 1

$$\eta_* = \frac{2}{H_0\sqrt{\Omega_{m0}}}\left[\sqrt{a_*+a_{eq}} - \sqrt{a_{eq}}\right], \tag{7.3}$$

where $a_* \simeq (1101)^{-1}$ is the scale factor at recombination, and the scale factor at matter-radiation equality is

$$a_{eq} = 4.15\cdot 10^{-5}\left(\Omega_{m0}h^2\right)^{-1}. \tag{7.4}$$

For the time conformal time today we will use

$$\eta_0 = \eta_* + \int_{a_*}^1 \frac{da}{a^2 H(a)}. \tag{7.5}$$

Here $H(a)$ is the Hubble factor given as

$$H = H_0\sqrt{\Omega_{m0}a^{-3} + \Omega_{de}a^{-3(1+\omega)}}. \tag{7.6}$$

We are thus assuming that dark energy did not play a major role at early times, but we are including it's effect in determining the conformal time today. Recall that we are employing units of $c = 1$, hence the value of the Hubble constant is

$$H_0 = 100h\,\mathrm{Mpc}^{-1}\mathrm{km\,s}^{-1} \simeq 3.33 \cdot 10^{-4} h\,\mathrm{Mpc}^{-1}. \tag{7.7}$$

Recall that $l_{eq} \equiv k_{eq}\eta_0$, where k_{eq} is the wavelength of the mode entering the horizon at equality. In chapter 4 we showed that it is given by $k_{eq} = \sqrt{\frac{\Omega_{m0}H_0^2}{a_{eq}}}$ which we can simplify

$$k_{eq} = \sqrt{\frac{\Omega_{m0}H_0^2}{a_{eq}}} = \sqrt{\frac{\Omega_{m0}H_0^2}{4.15 \cdot 10^{-5}\left(\Omega_{m0}h^2\right)^{-1}}} = 0.073\mathrm{Mpc}^{-1}\Omega_{m0}h^2. \tag{7.8}$$

This gives us a value of $l_{eq} \simeq 150 - 160$ for a standard ΛCDM model. We now turn to the Baryon-Photon density fraction R, which is given by

$$R = \frac{3}{4}\frac{\rho_b}{\rho_\gamma} \tag{7.9}$$

where the densities are

$$\rho_b = \Omega_b \rho_{cr} a^{-3} \tag{7.10}$$

$$\rho_\gamma = \rho_{cr}\frac{2.47 \cdot 10^{-5}}{h^2}\,a^{-4}, \tag{7.11}$$

where $\rho_{cr} = 1.88\,h^2 \times 10^{-29}\mathrm{g\,cm}^{-3}$ is the critical density. These equations can be used to simplify the expression for R as

$$\begin{aligned} R = \frac{3}{4}\frac{\rho_b}{\rho_\gamma} &= \frac{3}{4}\frac{\Omega_b a^{-3}}{\frac{2.47\cdot 10^{-5}}{h^2}\,a^{-4}} \\ &= 3.0364 \cdot 10^4 \Omega_b h^2 a. \end{aligned} \tag{7.12}$$

At recombination the typical value for R for a standard ΛCDM model is

$$R(\eta_*) \simeq 3.0364 \cdot 10^4 \Omega_b h^2 a_* \simeq 0.676. \tag{7.13}$$

Once we have the value for $R_* = R(\eta_*)$, we can calculate the sound horizon of the baryon-photon fluid which is

$$r_s(\eta) = \frac{2}{3\,k_{eq}}\sqrt{\frac{6}{R_{eq}}}\ln\left[\frac{\sqrt{R_* + R_{eq}} + \sqrt{1+R_*}}{1+\sqrt{R_{eq}}}\right]. \tag{7.14}$$

For the spectral amplitude δ_H we will use from [13]

$$\delta_H = 1.94 \cdot 10^{-5} \Omega_{m0}^{-0.785 - 0.05 \ln \Omega_{m0}}, \tag{7.15}$$

and from the same article find that the silk dampening wavelength can be approximated by

$$k_D \simeq 1.6(\Omega_b h^2)^{0.52} (\Omega_{m0} h^2)^{0.73} \left[1 + (10.4 \Omega_{m0} h^2)^{-0.95}\right] \mathrm{Mpc}^{-1}, \tag{7.16}$$

which gives $k_D \simeq 1600$ for a standard ΛCDM model.
This concludes our discussion of the numerics of the issue. We now have all we need to plot the small scale C_l. To do this we have written the code `caclPowerSpec.py` for a user friendly plotting procedure. A typical usage of the code is

```
python calcPowerSpec.py --omegab=0.07 --omegam=0.4\
  --omegaw=0.6 --lmin=300 etc...
```

The default values of in the code coincides with those of a standard ΛCDM model, hence supplying the code with no command line arguments will plot the power spectrum of a standard ΛCDM model with $\Omega_b = 0.05$, $\Omega_\Lambda = 0.7$, $\Omega_{m0} = 0.3$, $h = 0.7$ and $\omega = -1$. We will now move onwards to see what we can extract from our model.

7.2.1 Accuracy

We will not dwell too much on the question of accuracy at this time, but one thing is worth mentioning. The transfer functions we utilized in our method are strictly not valid for intermediate to large scales, i.e. they are only valid for $k \gg k_{eq}$ which translates into angular space to approximately $l \gg l_{eq} \simeq 160$. We therefore expect our solution to diverge around l_{eq}. Unfortunately the first acoustic peak is not far from this value, so we expect the first peak to be blurred out from the occurring divergencies appearing from the transfer functions. The natural way to fix this would of course be to use a transfer function valid on all scales. There does exists [13] a transfer function for baryons and cold dark matter valid on all scales, but these are quite complicated which will yield a corresponding raise in the complexity level in the final expression for the CMB. For our purposes, the gains in transparency in our model far outweigh the need for the inclusion of the first peak. A great deal of work has been done in the litterateur on the first peak, so we will concentrate on the other peaks instead.

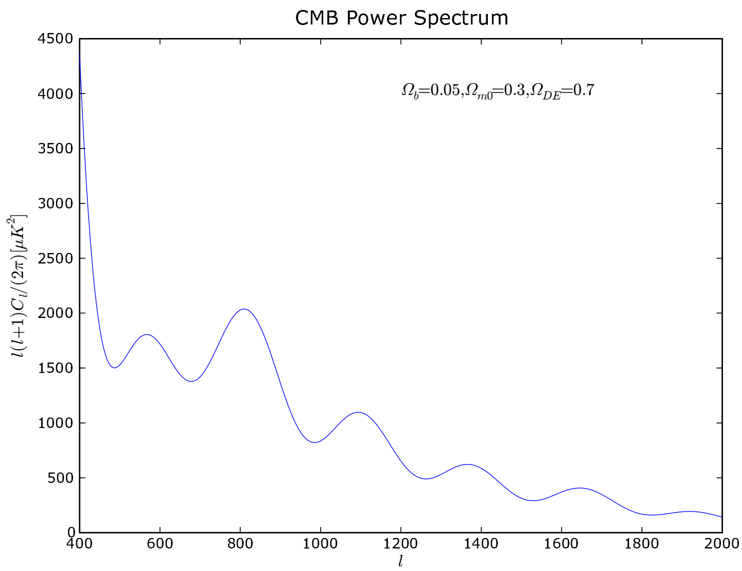

Figure 7.1: CMB power spectrum with $\Omega_b = 0.05$, $\Omega_\Lambda = 0.7$, $\Omega_{m0} = 0.3$, $h = 0.7$ and $\omega_{de} = -1$.

7.3 Peak Locations

In chapter 4, we discussed the possible locations of the peaks of the CMB power spectrum. These corresponded to the locations where $\cos(kr_s(\eta_*)) = \pm 1$, which in angular space would be

$$l_p = n\pi\eta_0/r_s(\eta_*). \tag{7.17}$$

This would be entirely true if the the projection from k-space to l-space, $k\eta_0 \to l$ was exact. This is because the spherical Bessel function $j_l(x)$ does not peak exactly at $x = l$, but at a slightly lesser value. For instance[10], the function $j_{100}(x)$ peaks at approximately $x \simeq 90$. As a first approximation we could then set $l \to l/0.9$ in our code to see the result. This turns our to be in a good correspondence with the power spectrum obtained by CMBFast or other codes. Hence a better approximation to the peaks would be

$$l_p = 0.9n\pi\eta_0/r_s(\eta_*). \tag{7.18}$$

Figure 7.1 shows the power spectrum for a standard ΛCDM model. In a flat cosmology, the distance between the peaks is more or less constant. The peak locations are sensitive to Ω_{m0} and Ω_b, and to some extent Ω_Λ (which changes the value of η_0). From figure 7.2 we can see how the position of the second peak shifts to lower l as the matter content increase and to slightly higher l with increasing Ω_b. Observe that this graph also includes a change in Ω_Λ through $\Omega_{m0} = 1 - \Omega_\Lambda$.

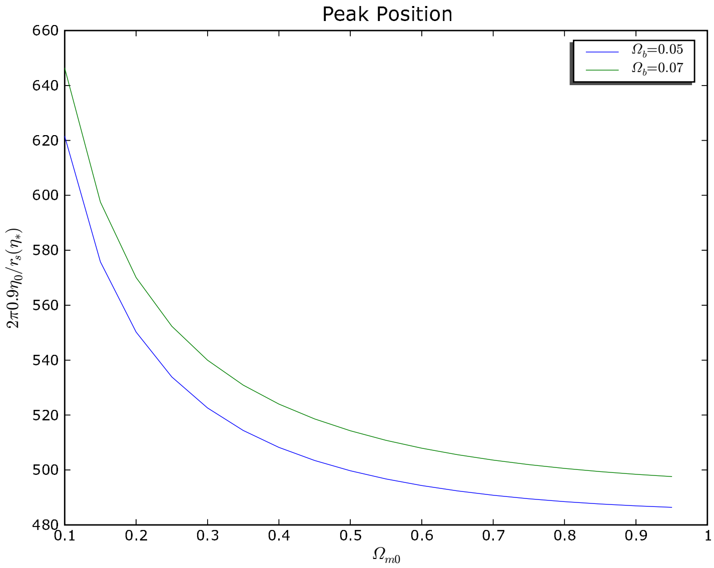

Figure 7.2: Location of the second peak in terms of a changing Ω_{m0}. The two curves represent two models with a differing baryon content of $\Omega_b = 0.05$ and $\Omega_b = 0.07$ respectively .

7.3.1 Peak Heights

We can first observe from figure 7.1 that where we would expect the C_l's to be zero (where $\cos(lr_s(\eta_*)/\eta_0) \simeq 0$), we see a trough instead. The reason for this is quite simply the effects of the dipole. Although smaller than the monopole, the dipole contributes the most to the C_l's where the monopole is 0. This changes the zero's of the CMB to a trough.
Concerning the peak heights, we can clearly see that our solution is a bit underestimated when comparing figure 7.1 to figure 7.4. The reason for this is twofold. Firstly we have assumed that only the highest peak of the spherical Bessel contributes to our integral. This is not entirely correct, as $j_l^2(x)$ has many peaks before $x \simeq l$. Although these peaks have a much lower amplitude than the one at $x \simeq l$, they do contribute a minute amount to the

final expression for the CMB.
The second reason for the underestimation of our solution is the concept of *Radiation driving*. It is caused by the decay of the gravitational potential $\tilde{\Phi}$ in the transition between radiation-dominated to matter-dominated epoch (see [37]). A decay of the gravitational potential causes the perturbations to rise in power, hence being called Radiation driving. This effect would give a rise in the power of the C_l. We have for simplicity ignored this effect in our derivation of the CMB, hence we have a noticeable lack of power in our result.

7.4 Baryon Signature

We can imagine the perturbations of the baryon-photon plasma as an oscillator with an effective mass of $(1+R)$ in our tight coupling approximation. The zeros of $\tilde{\Theta}_0$ is shifted by an amount of of $(1+R)\tilde{\Phi}$ (see equation (4.82)), and we can see that even and odd peaks get increased and suppressed respectively. This effect creates a characteristic effect in the CMB power spectrum known as *Baryon Suppression*. The effect would be non-existent in a cosmology with no baryons and therefore makes estimation of the baryon content easier than other cosmological parameters [14, 13].

As we can see from figure 7.3, the second and third peak get suppressed and increased respectively. There is also a slight shift in the peak locations due to larger baryon fraction. But observe that this dampening effect occurs only in these two peaks as Silk dampening becomes more significant at higher l-values, although we can see the fourth peak get's slightly more damped. See [5, 38] for more details on cosmological parameter dependence.

7.5 Integrated Sachs-Wolfe Effect

In our treatment here we have simplified our problem by not including the integrated Sachs-Wolfe term in the calculation of the CMB power spectrum. We will here mention some of the aspects of the ISW, without going into too much of the details. Recall that that the effect was quantified by the integral

$$\tilde{\Theta}_l^{ISW} \simeq \int_0^{\eta_0} d\eta \, e^{-\tau} \left(\dot{\tilde{\Psi}} - \dot{\tilde{\Phi}} \right) j_l \left(k(\eta_0 - \eta_*) \right). \tag{7.19}$$

The ISW quantifies the shift in energy of the photons from time varying gravitational potentials after recombination. In the standard ΛCDM model

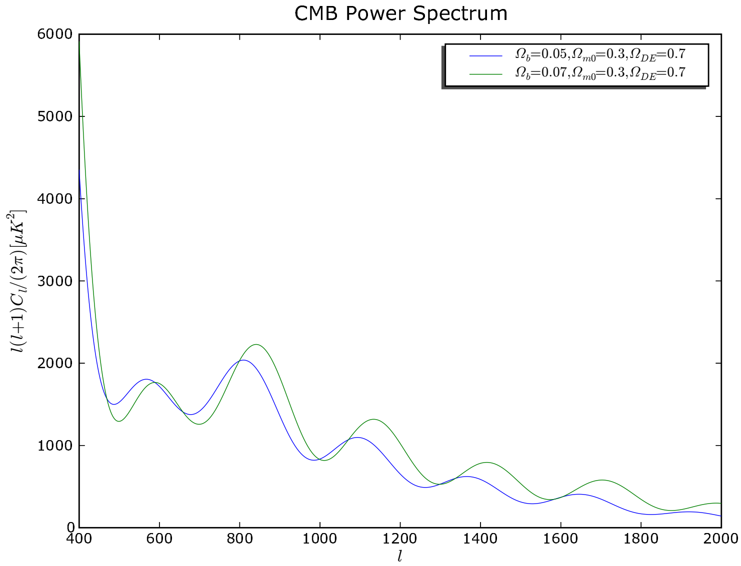

Figure 7.3: Recovering baryon signature from the CMB. The two curves represent two models with a differing baryon content of $\Omega_b = 0.05$ and $\Omega_b = 0.07$ respectively .

there is a slight decay of Ψ because of the remaining radiation after last scattering, in addition to a change induced by dark energy at later times. It has therefore natural to split up these effects into an *early ISW* and a *late ISW*. The early ISW effects the higher multipoles[10] l and can be approximated roughly by

$$\tilde{\Theta}_l^{early-ISW} \simeq 2(\tilde{\Phi}(\eta_*) - \tilde{\Phi}(\eta_0))j_l\,(k\eta_0)\,, \tag{7.20}$$

which adds coherently with the dipole and monopole, since they have the same bessel function dependence. This gives a slight increase in power at higher l-values[21]. This also explains some of the under-estimation of our analytic model.

The late ISW is sensitive to the dark energy[9] component of the universe. Since Λ domination is a recent event, only modes entering the horizon at later times are affected, i.e at large scales. These modes will get a boost in power, tilting the SW-plateau from section 6.4 to some extent[7]. The interested reader can in [11] see more about the effects of more exotic dark energy models on the CMB power spectrum.

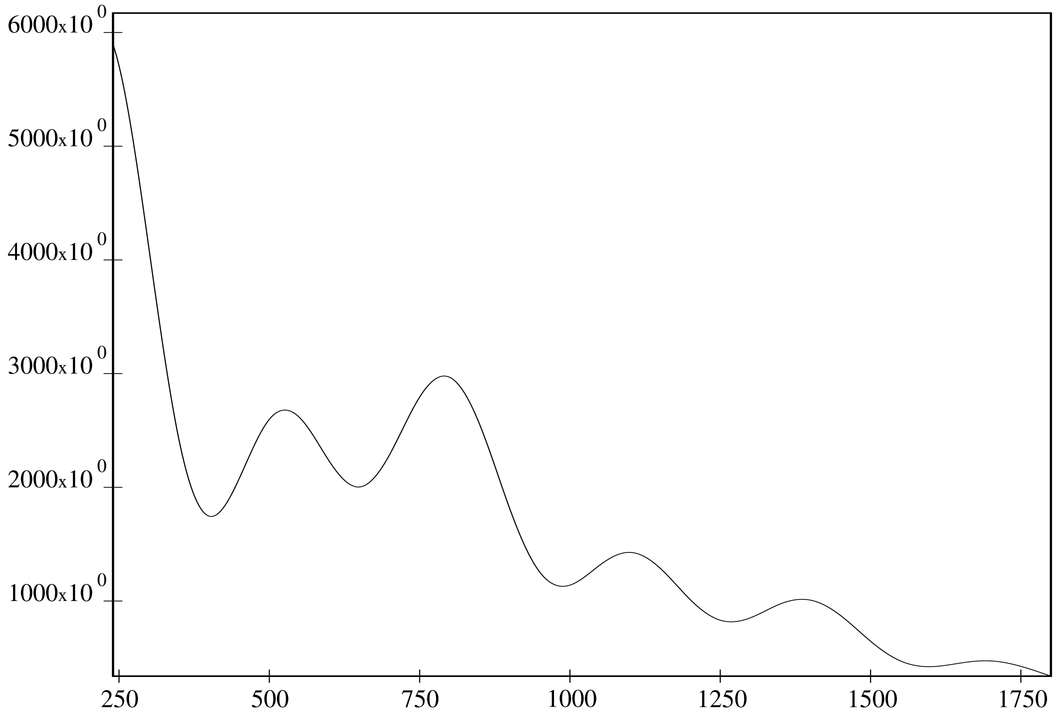

Figure 7.4: The CMB power spectrum obtained from CMBEasy. The curve is for a standard ΛCDM model with $\Omega_b = 0.05$, $\Omega_{m0} = 0.3$ and $\Omega_\Lambda = 0.7$.

7.6 Conclusion

In the previous sections we have seen how our simplified model works to reconstruct the small scale CMB power spectrum. Considering all the approximations we have done throughout our calculations, our model is in good accordance with the CMBEasy result. As mentioned earlier, the lack of inclusion of the first peak comes from the Transfer functions utilized which diverge when $l \to l_{eq}$. Hence a first improvement of our model would be to use a Transfer function valid on all scales, which one can find in [13]. Another rectification of our work on small scales would be to include the effects of the decay of $\tilde{\Phi}$ during radiation domination. This would increase the amplitude on intermediate to small scales[24], an effect that comes from the fact that radiation inhibits clumping. Adding a decaying $\tilde{\Phi}$ in our setting would mean to return to the formal solution of the monopole (equation (4.73)) and split up the integral into a radiation part (from $\eta \simeq 0$ to $\eta \simeq \eta_{eq}$), and a

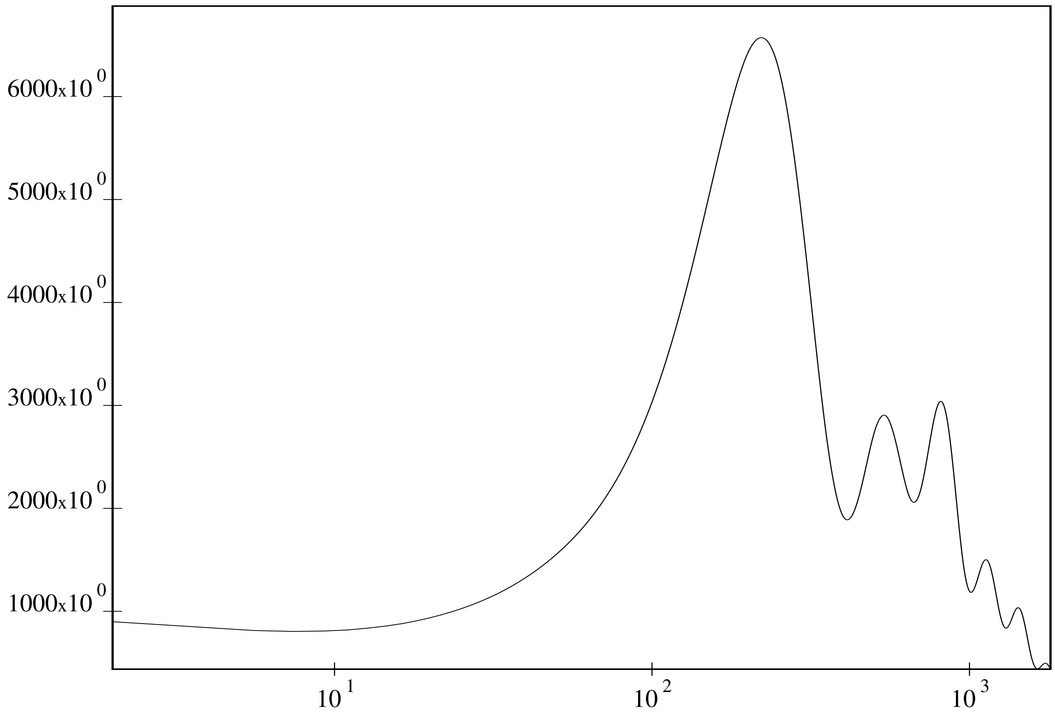

Figure 7.5: The full CMB power spectrum obtained from CMBEasy for all l. The curve is for a standard ΛCDM model with $\Omega_b = 0.05$, $\Omega_{m0} = 0.3$ and $\Omega_\Lambda = 0.7$.

matter part (from $\eta \simeq \eta_{eq}$ to $\eta \simeq \eta_*$). For the gravitational potential during radiation domination we would use from [37]

$$\Phi_{rad} \simeq 3\frac{\tilde{\Phi}(0)}{(c_s\eta)^3}\left[\sin(c_s\eta) - c_s\eta\cos(c_s\eta)\right] \tag{7.21}$$

But of course, for realistic models including effects such as reionization and neutrinos, we would have to solve our problem numerically. To complete the picture we have drawn of the CMB anisotropies we would also have to include the effect of polarization which we unfortunately did not cover. For a review of CMB polarization see[23].
With the work we have done in this thesis we are able to understand much better how the "black box" numerical codes work. But more importantly we have developed the tools necessary to write codes for non-standard cosmological models like anisotropic and inhomogenous universes. By the time

of this writing [26] seem to indicate an anisotropically expanding universe. Hence to get an accurate description of the power spectrum in this setting we would have to return to the Boltzmann equation from chapter 2 and redo much of our calculations.

The present work represents an effort to present in a physically transparent manner the theory of the anisotropies of the Cosmic Microwave Background radiation.

Chapter 8

Epilog

Some final comments are in order at this time. As mentioned earlier I have presented the current cosmological paradigm on the CMB temperature fluctuations. One might ask if what I have added to our field of research. Through out this work I have tried to find alternate ways (from the literature) on doing things. In chapter 2, I opted to do the full calculation of the Boltzmann equation in conformal time, where in the literature this is done in cosmic time and conformal time is introduced at the end. In chapter 3, I made an interesting discovery that one can obtain the expression for the perturbation to the gravitational potential from inflation without using the Einstein equations. And maybe more importantly, the small scale solution of the CMB power spectrum in chapter 6 is new. But as mentioned in the conclusion there are many ways to improve this model. In future work I would especially like to do a full calculation of the CMB power spectrum for an anisotropic universe.

Appendix A

The Boltzmann Equation

A.1 The Boltzmann Equation

We will in this section derive[17] the Boltzmann equation which is an integral part of a detailed study of the CMB fluctuations. We begin by defining the distribution function in phase space $f(t, \mathbf{r}, \mathbf{v})$ for a monoatomic particle ensemble consisting of N particles. The phase space here is the configuration s pace coordinates $\mathbf{r} = (x_1, x_2, x_3)$ and the velocity field $\mathbf{v} = (v_1, v_2, x_3)$. With the time parameter t, the distribution function is a function of 7 variables, which we assume to be independent.
Let $d^3r\, d^3v$ be the volume element in phase space cantered about position r. The number of particles in the volume element at time t and velocity in the range $\mathbf{v}$ and $\mathbf{v} + d^3v$ is given by

$$dN = f(t, r, v) d^3r\, d^3v\,. \tag{A.1}$$

In this derivation we are assuming that the acceleration $\mathbf{a}(t, r, v)$ imposed on the molecules by external forces are divergence free in velocity space, i.e. $\nabla_{\mathbf{v}} \cdot \mathbf{a}(t, r, v) = 0$. This is the case for most (external) force fields including gravity and the Lorentz force.
Assuming (for now) no intermolecular collisions, we let $t \to t' = t + dt$ with $dt \ll 1$. Thus we will working in the linear regime in dt in all the following calculations. The particles in the original phase space element would now be found in thee volume element d^3r' about the spatial location $r' = r + vdt$. They would acquire new velocity in the range $v' = v + a(t, r, v)\, dt$ and $v' + d^3v'$. The number of particles in this phase space element would now be

$$dN' = f(t + dt, r + vdt, v + adt) d^3r'\, d^3v'. \tag{A.2}$$

We want to find a relation between the new phase space element $d^3r'\, d^3v'$ and the original one $d^3r\, d^3v$. Recall that volume elements between two coordinate systems are related by the Jacobian of the transformation by

$$d^3r'\,d^3v' = |J|\,d^3r\,d^3v\,, \tag{A.3}$$

where $|J|$ is the determinant of the Jacobian Matrix given by

$$J = \frac{\partial(r', v')}{\partial(r, v)} = \begin{bmatrix} \frac{\partial x'_i}{\partial x_j} & \frac{\partial v'_i}{\partial x_j} \\ \frac{\partial x'_i}{\partial v_j} & \frac{\partial v'_i}{\partial v_j} \end{bmatrix} = \begin{bmatrix} [\delta_{ij}] & [\partial_j a_i dt] \\ [\delta_{ij} dt] & [\delta_{ij} + \partial_{v_j} a_i dt] \end{bmatrix}.$$

Here, $\partial_j = \frac{\partial}{\partial x_j}$ and $\partial_{v_j} = \frac{\partial}{\partial v_j}$. The brackets denotes a matrix with the indicated elements. Calculating the determinant of J up to linear terms in dt gives

$$|J| = 1 + \sum_{i=1}^{3} \partial_{v_j} a_i\,dt + (\text{...higher order terms...})\,. \tag{A.4}$$

Remembering that the acceleration field is divergence free in velocity space, we get

$$|J| = 1\,.$$

Thus the phase space volume element remains unchanged throughout the displacement in phase space. In the lack of collision effects, the particles in $d^3r\,d^3v$ at time t, would be found in $d^3r'\,d^3v'$ at time $t + dt$. Hence

$$dN' = dN \qquad (\text{...no collisions...})\,. \tag{A.5}$$

If we now add scattering effects, the number of particles in the phase space element may change,

$$dN' - dN = dN_{collision} \qquad (\text{...with collisions...})\,, \tag{A.6}$$

where $dN_{collision}$ is the net flow of particles in/out of the phase element. Inserting equations (A.1) and (A.2) in equation (A.6) we get

$$[f(t + dt, r + vdt, v + adt) - f(t, r, v)]\,d^3r\,d^3v = \frac{\delta f(t, r, v)}{\delta t} d^3r\,d^3v\,dt\,, \tag{A.7}$$

where we have defined

$$dN_{collision} \equiv \delta f(t, r, v) d^3r\,d^3v \;= \frac{\delta f(t, r, v)}{\delta t} d^3r\,d^3v\,dt\,. \tag{A.8}$$

The expression $\frac{\delta f(t,r,v)}{\delta t}$ is at this point just formally introduced, which at a later time will be replaced by an integral (collision integral).

Equation (A.8) is valid for all phase space volumes $d^3r\, d^3v$. Hence, after division by dt we obtain

$$\frac{f(t+dt, r+vdt, v+adt) - f(t,r,v)}{dt} = \frac{\delta f(t,r,v)}{\delta t}\,. \tag{A.9}$$

Recall that the LHS is the (total) time derivative of a multivariable function , thus we get

$$\frac{df}{dt} = \frac{\delta f(t,r,v)}{\delta t} \equiv C[f]\,, \tag{A.10}$$

where the RHS is often referred to as the collision term. By the preceding line of argument we have proven

$$\frac{df}{dt} = C[f]\,. \tag{A.11}$$

This is The Boltzmann Equation, which is valid in both equilibrium and non-equilibrium conditions, and serves as a basis for our work.

Appendix B

Numerical Codes

Here I will present some of the computer codes used in this thesis. For some of the simulations I have opted to use the Python scripting language as numerical schemes are easy and fast to implement in this language. For efficiency I have used either the NumPy or the mathlibplot package which is included in most python installments or can be acquired for free.

B.1 The Growth Function

In chapter 5 we encountered the growth function which was defined by an integral. The following code implements a numeric scheme (specifically the Trapezium rule) to integrate the function given by equation (5.45).

```
 #!/usr/bin/env python
 #calcNumericInt.py
 # Code for calculating functions defined as a function multiplied by
 #an integral,writing the result to file .Function is of the form
 #F(xmax)=factor(xmax) \int_xmin^xmaxfunction(x') dx'

import sys, re, os from NumPy import *

try:
    factor=sys.argv[1]
    func=sys.argv[2]
    text=sys.argv[3]
    interval=sys.argv[4]
    dx=float(sys.argv[5])
```

```
except:
    usage="Usage: %s  'factor(x)' 'function(x)' 'header text' \
        '[xmin,xmax]' dx " % sys.argv[0]
    print usage; sys.exit(1)

intervalpattern=r"([+\-]?\d\.\d+[Ee][+\-]\d\d?|[+\-]?\d+\.?\d*|
                    [+\-]?\.\d+)"
numbs = [ float(x) for x in re.findall(intervalpattern,interval)]
xmin=numbs[0] xmax=numbs[1]

def funct(x):
    return eval(func)

def funct2(x):
    return eval(factor)

def trap(min,max):
    n=40
    h=(max-min)/(n-1)
    x=[ (min +z*(h))  for z in range(0,n)]
    x=array(x)
    #print x
    #print funct(x)
    integral = (h/2* ( funct(x[0])+funct(x[n-1]) +
                2*sum(funct(x)[1:n-1])))
    #print integral
    return integral

h2=int((xmax-xmin)/dx)

xv=[(xmin +dx*num) for num in  range(0,h2+1)  ] xv=array(xv)
yv=[(funct2(xv[i])*trap(xmin,xv[i])) for i in range(0,h2+1)]

filen=text+'.dat' ofile = open(filen, 'w')
ofile.write(text+"\n")
for i in range(0,h2+1):
    ofile.write("%f %f \n"% (xv[i],yv[i]))

ofile.close()
```

This next code plots data output from `calcNumericInt.py` script and saves the plot to a ps file. The plots in figure (ref) is created by this program.

```
 #!/usr/bin/env python
 # plotNumInt.py
 # Code for plotting sets of data and saving output
  import sys, re, os,time

from NumPy import *

import Gnuplot

filename=sys.argv[1]

ifile = open(filename, 'r')#open file for reading

file=ifile.readlines()

ifile.close()
x=[]
y=[]
filename=file[0][:-1]#strip new line

file=file[1:]

for line in file:
    line=line.split()
    x.append(float(line[0]))
    y.append(float(line[1]))

if(len(sys.argv)>2):
    filename2=sys.argv[2]
    ifile = open(filename2, 'r') # open file for reading

    file=ifile.readlines()

    ifile.close()
    x2=[]
    y2=[]
    filename2=file[0][:-1]
    file=file[1:]
```

```
    for line in file:
        line=line.split()
        x2.append(float(line[0]))
        y2.append(float(line[1]))

if(len(sys.argv)>3):
    filename3=sys.argv[3]
    ifile = open(filename3, 'r') # open file for reading

    file=ifile.readlines()

    ifile.close()
    x3=[]
    y3=[]
    filename3=file[0][:-1]
    file=file[1:]
    for line in file:
        line=line.split()
        x3.append(float(line[0]))
        y3.append(float(line[1]))


g = Gnuplot.Gnuplot(persist=1)#,debug=1)
 #persist=1: let plot remain on the screen
#g('set pointsize 2') g('set data style lines')
d1 = Gnuplot.Data(x,y, title='%(filename)s' % vars())
if(len(sys.argv)==3):
    d2 = Gnuplot.Data(x2,y2, title='%(filename2)s' % vars())
elif(len(sys.argv)==4):
    d2 = Gnuplot.Data(x2,y2, title='%(filename2)s' % vars())
    d3 = Gnuplot.Data(x3,y3, title='%(filename3)s' % vars())
if(len(sys.argv)==3):
    g.plot( d1,d2,xlabel='Expansion Factor a',
            ylabel='Growth Function D(a)')
elif(len(sys.argv)==4):
    g.plot( d1,d2,d3,xlabel='Expansion Factor a',
            ylabel='Growth Function D(a)')
else:
    g.plot(d1,xlabel='Expansion Factor a',
            ylabel='Growth Function D(a)')
```

```
 g.hardcopy(filename='case.eps', enhanced=1,
   color=1, fontname='Times-Roman', fontsize=28)
 time.sleep(3)
```

B.2 The CMB Power Spectrum

In this section we present the code used to simulate and plot the CMB power spectrum , specifically equation (6.71) from chapter 6.

```
#!/usr/bin/env python
# calcPowerSpectrum.py

 # Code for calculating the CMB power spectrum

import sys, re, os,getopt,time

from scipy import integrate

from pylab import *

alpha=1.0
 lmin=150 lmax=1500 omegam=0.3 omegab=0.05 omegaw=0.7
w=-1.0 h=0.7 arec=1.0/1101 case='powspec1' approx=1 Yp=0.23

def setDefault():
    return 0.3,0.05,0.7,-1.0,0.7,1.0

options, args = getopt.getopt(sys.argv[1:],
'',['xsi=','phase=','beta=','alpha=','lmin=','lmax=','omegam='\
    ,'omegab=','omegaw=','w=','h=','arec=','case=','approx='])

for option, value in options:
    if option in ('--alpha'):
        alpha=float(value)
```

```
    elif option in ('--lmin'):
        lmin=float(value)
    elif option in ('--lmax'):
        lmax=float(value)
    elif option in ('--omegam'):
        omegam=float(value)
    elif option in ('--omegab'):
        omegab=float(value)
    elif option in ('--omegaw'):
        omegaw=float(value)
    elif option in ('--w'):
        w=float(value)
    elif option in ('--arec'):
        arec=float(value)
    elif option in ('--case'):
        case=value
    elif option in ('--approx'):
        approx=int(value)
    elif option in ('--beta'):
        beta=float(value)

# The baryon to photon density ratio
def R(a):
    return (30364.0*omegab *pow(h,2)*a)
# The Hubble factor without H_0 def H(a):
    return (pow(omegam*pow(a,-3)+omegaw*pow(a,-3.0*(1+w)),0.5))

keq=0.073*omegam *pow(h,2) aeq=4.15*pow(10,-5.0)/(omegam *pow(h,2))
eta_rec=6000.0/(h*pow(omegab,0.5))*(pow(arec+aeq,0.5)-pow(aeq,0.5))
eta_eq=6000.0/(h*pow(omegab,0.5))*(pow(aeq+aeq,0.5)-pow(aeq,0.5))
#The integrand of the conformal time integral
def f2(a):
    return (1.0/(a*a*H(a)))
eta_0=eta_rec+3000.0/h*integrate.quad(f2,arec,1)[0]

leq=eta_0*keq

k_D=1.6*pow(omegab*h*h,0.52)*pow(omegam*h*h,0.73)\
    *(1+pow(10.4*omegam*h*h,-0.95))
lD=k_D*eta_0 a4=-0.0473*pow(omegam*pow(h,2),-0.0639)
a3=1.03*pow(omegam*pow(h,2),0.0335)
```

```
m=a3*pow(omegab*pow(h,2),a4)*pow(1+pow(omegab*pow(h,2),1.8),0.2)
Req=1.2060106*omegab/omegam
 l=arange(lmin,lmax,1.0)
# Growth function
def Gf(a):
    return (5.0/2.0*omegam*H(a)*\
            integrate.quad(lambda x:(1.0/pow(x*H(x),3)),0.0,a)[0])
#Silk dampening function D_l
def D(l):
    return (exp(-pow(l/lD,m)))

# The modified sound horizon
Zr=2.0/3.0*pow(6.0/Req,0.5)*(log(pow(1+R(arec),0.5)+pow(R(arec)+Req,0.5))\
    -log(1+pow(Req,0.5)))
# Estimation of the spectral amplitude d_H from C_10 from COBE
dH=1.94*pow(10,-5)*pow(omegam,-0.785-0.05*log(omegam))/pow(25,0.5)

clconst=2*2.7*2.7*pow(10,12)*pi/2.0*pow(omegam*dH/Gf(1),2)

arec/aeq*(-6.0*pow(1+arec/aeq,0.5)+(2.0+3.0*arec/aeq)*\
    log((pow(1+arec/aeq,0.5)+1)/(pow(1+arec/aeq,0.5)-1)))

#The power spectrum

def Cl(x):
    x=x/0.9
    sigma=(1-omegab/omegam)
    nu=omegab/omegam
    g=Ds(x)*arec/aeq*(-6.0*pow(1+arec/aeq,0.5)+(2.0+3.0*arec/aeq)*\
        log((pow(1+arec/aeq,0.5)+1)/(pow(1+arec/aeq,0.5)-1)))
    part11=Ds(x)*Ds(x)*pow(1+R(arec),-0.5)*pow(cos(alpha*x/leq*Zr),2)
    part12=(Ds(x)*R(arec)*pow(1+R(arec),-0.25)*\
            4.0/3.0*cos(alpha*x/leq*Zr))\
        *(12.0*sigma*2.0/3.0*pow(leq/alpha,2)/(x*x)*log(beta*x/(8*leq))+\
            1.0/4.0*pi*pow(leq/(alpha*x),1)*2.07*nu*pow(1+R(arec),-0.75)*\
                g*sin(alpha*x/leq*Zr) )
    part13=(4.0/9.0*pow(R(arec),2))*(144.0*8.0/15.0*\
            pow(leq/alpha,4)/((x+3)*(x+2.0)*(x-1.0)*(x-2.0))*\
                \pow(log(beta*x/(8*leq)),2)*\
            sigma*sigma+24*nu*sigma*3.0/16.0*pi*pow(log(beta*x/(8*leq)),1)*\
            pow(leq/(alpha*x),3)*2.07*pow(1+R(arec),-0.75)*\
```

```
        g*pow(sin(alpha*x/leq*Zr),1)+\
        2.07*2.07*2.0/3.0*pow(leq/(alpha*x),2)*\
        pow(g*nu*sin(alpha*x/leq*Zr)*nu,2)*pow(1+R(arec),-1.5) )
    part1=part11+part12+part13
    part21=0.0#pi*pow(1+R(arec),-1)*pow(3.0,0.5)*sin(2.0*alpha*x/leq*Zr)
    part22=x*(x+1.0)/( (2.0*x+1.0)*(2.0*x-1.0))-\
        2.0*x*pow(x+1.0,2)/((2.0*x+3.0)*(2.0*x+1.0)*(2.0*x-1.0))
    part31=3*Ds(x)*Ds(x)*pow(1+R(arec),-1.5)*pow(sin(alpha*x/leq*Zr),2)
    part32=-1.0/3.0*(x+1.0)/(x-1.0)+2.0/3.0*pow(x+1.0,2)/((x+2.0)*(x-1.0))
    return (part1+part21*part22+part31*part32)

y=clconst*Cl(l)
S1=r'$\Omega_b=%s , \Omega_{m0}=%s , \Omega_{DE}=%s $' \
    %(omegab,omegam,omegaw)
omegam,omegab,omegaw,w,h,xsi=setDefault()

keq=0.073*omegam *pow(h,2)
aeq=4.15*pow(10,-5.0)/(omegam *pow(h,2))
eta_rec=6000.0/(h*pow(omegab,0.5))*(pow(arec+aeq,0.5)-pow(aeq,0.5))
eta_eq=6000.0/(h*pow(omegab,0.5))*(pow(aeq+aeq,0.5)-pow(aeq,0.5))
#The integrand of the conformal time integral
def f2(a):
    return (1.0/(a*a*H(a)))
eta_0=eta_rec+3000.0/h*integrate.quad(f2,arec,1)[0]

leq=eta_0*keq

k_D=1.6*pow(omegab*h*h,0.52)*pow(omegam*h*h,0.73)*\
    (1+pow(10.4*omegam*h*h,-0.95))
lD=k_D*eta_0 a4=-0.0473*pow(omegam*pow(h,2),-0.0639)
a3=1.03*pow(omegam*pow(h,2),0.0335)
m=a3*pow(omegab*pow(h,2),a4)*pow(1+pow(omegab*pow(h,2),1.8),0.2)
Req=1.2060106*omegab/omegam

l=arange(lmin,lmax,1.0) #l=r_[lmin:lmax+1:1.0] # Growth function def
Gf(a):
    return (5.0/2.0*omegam*H(a)*integrate.quad(lambda
        x:(1.0/pow(x*H(x),3)),0.0,a)[0])

#Silk dampening function D_l
def Ds(l):
```

```
    return (exp(-pow(l/lD,m)))

# The modified sound horizon
Zr=2.0/3.0*pow(6.0/Req,0.5)*(log(pow(1+R(arec),0.5)+\
    pow(R(arec)+Req,0.5))-log(1+pow(Req,0.5)))

# Estimation of the spectral amplitude d_H from C_10 from COBE
dH=1.94*pow(10,-5)*pow(omegam,-0.785-0.05*log(omegam))/pow(25,0.5)

clconst=2*2.7*2.7*pow(10,12)*pi/2.0*pow(omegam*dH/Gf(1),2)

S2=r'$\Omega_b=%s , \Omega_{m0}=%s , \Omega_{DE}=%s $'
        %(omegab,omegam,omegaw)
y2=clconst*Cl12(l)

plot(l,y2,l,y)

legend((S2, S1),'upper right', shadow=True)

#legend((S1),'upper right', shadow=True)

title("CMB Power Spectrum")

ylabel(r"$l(l+1) C_l /(2\pi) [\mu K^2]$")
xlabel(r"$ l$")
#text(lmax-(lmax-lmin)/2.0,0.9*max(y),S1) show()
```

B.3 The Sound Horizon

Here we give the script used to plot the inverse sound horizon in chapter 7.

```
#!/usr/bin/env python # plotSoundHorizon.py

# Code for plotting the inverse sound horizon

import sys, re, os,getopt,time
```

```
from scipy import*
from pylab import *

alpha=1.0
 lmin=150 lmax=1500 omegam=0.3 omegab=0.05 omegaw=0.7
w=-1.0 h=0.7 arec=1.0/1001 case='powspec1' approx=1 Yp=0.23

options, args = getopt.getopt(sys.argv[1:],
'',['xsi=','phase=','beta=','alpha=','lmin=',
    'lmax=','omegam=','omegab=','omegaw=','w=',
    'h=','arec=','case=','approx='])
#print options #print args


for option, value in options:
    if option in ('--alpha'):
        alpha=float(value)
    elif option in ('--lmin'):
        lmin=float(value)
    elif option in ('--lmax'):
        lmax=float(value)
    elif option in ('--omegam'):
        omegam=float(value)
    elif option in ('--omegab'):
        omegab=float(value)
    elif option in ('--omegaw'):
        omegaw=float(value)
    elif option in ('--w'):
        w=float(value)
    elif option in ('--arec'):
        arec=float(value)
    elif option in ('--case'):
        case=value
    elif option in ('--approx'):
        approx=int(value)
    elif option in ('--beta'):
        beta=float(value)

# The baryon to photon density ratio
def R(a):
    return (30364.0*omegab *pow(h,2)*a)
```

```
# The Hubble factor without H_0 def H(a):
    return (pow(omegam*pow(a,-3)+omegaw*pow(a,-3.0*(1+w)),0.5))

keq=0.073*omegam *pow(h,2) aeq=4.15*pow(10,-5.0)/(omegam *pow(h,2))
eta_rec=6000.0/(h*pow(omegab,0.5))*(pow(arec+aeq,0.5)-pow(aeq,0.5))
eta_eq=6000.0/(h*pow(omegab,0.5))*(pow(aeq+aeq,0.5)-pow(aeq,0.5))

#The integrand of the conformal time integral
def f2(a):
    return (1.0/(a*a*H(a)))
eta_0=eta_rec+3000.0/h*integrate.quad(f2,arec,1)[0]

l=arange(0.1,1.0,0.05)

Growth function def Gf(a):
    return (5.0/2.0*omegam*H(a)*\
        integrate.quad(lambda x:(1.0/pow(x*H(x),3)),0.0,a)[0])
Gf2=vectorize(Gf)

def f2(a):
        return (1.0/(a*a*H(a)))
def f3(a,b):
    def f(z):return (1.0/(z*z*(pow(b*pow(z,-3)+\
                        (1-b)*pow(z,-3.0*(1+w)),0.5))))
    return integrate.quad(f,a,1)[0]

f32=vectorize(f3)

def Func(x):
    z=x
    keq=0.073*x*pow(h,2)
    aeq=4.15*pow(10,-5.0)/(x*pow(h,2))
    eta_rec=6000.0/(h*pow(omegab,0.5))*(pow(arec+aeq,0.5)-pow(aeq,0.5))
    eta_eq=6000.0/(h*pow(omegab,0.5))*(pow(aeq+aeq,0.5)-pow(aeq,0.5))
    eta_0=eta_rec+3000.0/h*f32(arec,x)
    leq=eta_0*keq
    Req=1.2060106*omegab/x
    Zr=2.0/3.0*pow(6.0/Req,0.5)*(log(pow(1+R(arec),0.5)+\
        pow(R(arec)+Req,0.5))-log(1+pow(Req,0.5)))
    return (2*0.9*leq/Zr*pi)
```

```
y=Func(l)
S1=r'$\Omega_b=%s $' %(omegab)
omegab=0.07
y2=Func(l)
S2=r'$\Omega_b=%s  $' %(omegab)

title("Peak Position ")
ylabel(r"$2\pi 0.9\eta_0/r_s(\eta_*) $")
xlabel(r"$ \Omega_{m0}$")

plot(l,y,l,y2)
legend((S1, S2),'upper right', shadow=True)
show()
```

Appendix C

Mathematical Supplement

In this part we present some of the mathematics required throughout this work.

C.1 A short note on Hankel Functions

Given a function $u(z)$ that satisfies the modified Bessel equation

$$u'' + \frac{1-2\alpha}{z}u' + \left[\left(\beta\gamma z^{\gamma-1}\right)^2 + \frac{\alpha^2 - \nu^2\gamma^2}{z^2}\right]u = 0 \tag{C.1}$$

Where all the Greek letters are parameters. The solution of this equation can written as

$$u = z^{\alpha} Z_{\nu}\left(\beta z^{\gamma}\right) \tag{C.2}$$

where $Z_\nu(z) = C_1 J_\nu(z) + C_2 J_{-\nu}(z)$. Here $J_\nu(z)$ is the ordinary Bessel function. Instead of using the ordinary Bessel function as solution, one can instead express the solution in the Hankel Functions of 1. and 2. kind $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$. We will use these functions instead because they have a simpler asymptotic behaviour. Hence our solution can be expressed as

$$Z_\nu(z) = c_1 H_\nu^{(1)}(z) + c_2 H_\nu^{(2)}(z) \tag{C.3}$$

The asymptotic limit of the Hankel functions for large z is

$$H_\nu^{(1)}(z) \simeq \sqrt{\frac{2}{\pi z}} e^{i\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)} \tag{C.4}$$

$$H_\nu^{(2)}(z) \simeq \sqrt{\frac{2}{\pi z}} e^{-i\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)} \tag{C.5}$$

For small z we will only need $H_\nu^{(1)}(z)$. This is

$$H_\nu^{(1)}(z) \simeq \frac{2^\nu}{i\pi}\Gamma(\nu)z^{-\nu} \tag{C.6}$$

If we now choose

$$\alpha = \frac{1}{2} \quad \gamma = 1 \quad \beta = k \quad z = -\eta \tag{C.7}$$

We see that equation (C.1) reduces to

$$u'' + \left(k^2 + \frac{1}{\eta^2}\left(\frac{1}{4} - \nu^2\right)\right) u = 0 \tag{C.8}$$

Which is exactly equation (3.71) in section 3.7.3. The solution of this equation is given by

$$u = \sqrt{-\eta}\, Z_\nu\left(-k\eta\right) \tag{C.9}$$

C.2 Short Note on Spherical Harmonics

The functions that are known as spherical harmonics are solutions of the eigenvalue problem

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) Y_{lm}(\theta,\phi) = -l(l+1)Y_{lm}(\theta,\phi). \tag{C.10}$$

The Y_{lm}'s form a complete set of orthogonal eigenfunctions satisfying

$$\int d\Omega Y_{lm}(\bar{n})Y_{l'm'}(\bar{n}) = \delta_{ll'}\delta_{mm'}, \tag{C.11}$$

where $d\Omega$ is the solid angle element subtended by the unit vector $\bar{n}$ with components

$$n_x = \sin\theta\cos\phi \tag{C.12}$$
$$n_y = \sin\theta\sin\phi \tag{C.13}$$
$$n_z = \cos\theta \tag{C.14}$$

The first few spherical harmonics are

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \tag{C.15}$$

$$Y_{10}(\theta,\phi) = i\sqrt{\frac{3}{4\pi}}\cos\theta \tag{C.16}$$

$$Y_{1,\pm1}(\theta,\phi) = \mp i\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi} \tag{C.17}$$

One important formula is the relation with the Legendre functions given by

$$P_l(\bar{n}\cdot\bar{n}') = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\bar{n})Y^*_{l'm'}(\bar{n}') \tag{C.18}$$

This formula can be "inverted" to obtain $Y_{lm}(\bar{n})$ in terms of Legendre functions. If we multiply the above equation by $d\Omega Y_{l'm'}(\bar{n}')$ and integrate, we get

$$\begin{aligned}\int d\Omega Y_{l'm'}(\bar{n}')P_l(\bar{n}\cdot\bar{n}') &= \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\bar{n})\int d\Omega Y_{l'm'}(\bar{n}')Y^*_{l'm'}(\bar{n}')\\ &= \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\bar{n})\delta_{ll'}\delta_{mm'}\\ &= \frac{4\pi}{2l+1}Y_{lm'}(\bar{n}')\delta_{ll'},\end{aligned} \tag{C.19}$$

where we have used the orthogonality condition (C.11). We hence obtain

$$\int d\Omega Y_{l'm}(\bar{n}')P_l(\bar{n}\cdot\bar{n}') = \delta_{ll'}\frac{4\pi}{2l+1}Y_{lm}(\bar{n}') \tag{C.20}$$

C.3 Some Integrals of Bessel Functions

We will need some integrals of Bessel functions to be able to calculate the C_l's. For easy reference I will reproduce them here, including some simplifications of these formulae. Using the expression given in [18]

$$\int_0^\infty dx\frac{J_\mu J_\nu}{x^\lambda} = \frac{\Gamma(\lambda)\Gamma(\frac{\mu+\nu-\lambda+1}{2})}{2^\lambda\Gamma(\frac{-\nu+\mu+\lambda+1}{2})\Gamma(\frac{\nu+\mu+\lambda+1}{2})\Gamma(\frac{\nu-\mu+\lambda+1}{2})}, \tag{C.21}$$

where $\Gamma(z)$ is the gamma function. By choosing one-half integer values for μ and ν we get the following equations

$$\int_0^\infty dx\frac{J^2_{l+\frac{1}{2}}}{x^\lambda} = \frac{\Gamma(\lambda)\Gamma(l-\frac{\lambda}{2}+1)}{2^\lambda\Gamma^2(\frac{\lambda+1}{2})\Gamma(l+\frac{\lambda}{2}+1)} \tag{C.22}$$

$$\int_0^\infty dx\frac{J_{l+\frac{1}{2}}J_{l-\frac{1}{2}}}{x^\lambda} = \frac{\Gamma(\lambda)\Gamma(l-\frac{\lambda}{2}+\frac{1}{2})}{2^\lambda\Gamma(\frac{2+\lambda}{2})\Gamma(l+\frac{\lambda+1}{2})\Gamma(\frac{\lambda}{2})} \tag{C.23}$$

$$\int_0^\infty dx\frac{J^2_{l-\frac{1}{2}}}{x^\lambda} = \frac{\Gamma(\lambda)\Gamma(l-\frac{\lambda}{2})}{2^\lambda\Gamma^2(\frac{\lambda+1}{2})\Gamma(l+\frac{\lambda}{2})} \tag{C.24}$$

For $\lambda = 2m$ we can simplify further to get

$$\int_0^\infty dx \frac{J^2_{l+\frac{1}{2}}}{x^{2m}} = \frac{(2m-1)!(l-m)!}{2^{2m}\Gamma^2(m+\frac{1}{2})(l+m)!} \tag{C.25}$$

$$\int_0^\infty dx \frac{J_{l+\frac{1}{2}}J_{l-\frac{1}{2}}}{x^{2m}} = \frac{(2m-1)!\Gamma(l-m+\frac{1}{2})}{2^{2m}m!(m-1)!\Gamma(l+m+\frac{1}{2})} \tag{C.26}$$

$$\int_0^\infty dx \frac{J^2_{l-\frac{1}{2}}}{x^{2m}} = \frac{(2m-1)!(l-m-1)!}{2^{2m}\Gamma^2(m+\frac{1}{2})(l+m-1)!}. \tag{C.27}$$

For odd values of λ we obtain

$$\int_0^\infty dx \frac{J^2_{l+\frac{1}{2}}}{x^{2m+1}} = \frac{(2m)!\Gamma(l-m+\frac{1}{2})}{2^{2m+1}(m!)^2\Gamma(l+m+\frac{3}{2})} \tag{C.28}$$

$$\int_0^\infty dx \frac{J_{l+\frac{1}{2}}J_{l-\frac{1}{2}}}{x^{2m+1}} = \frac{(2m)!(l-m-1)!}{2^{2m+1}\Gamma(m+\frac{3}{2})\Gamma(m+\frac{1}{2})(l+m)!} \tag{C.29}$$

$$\int_0^\infty dx \frac{J^2_{l-\frac{1}{2}}}{x^{2m+1}} = \frac{(2m)!\Gamma(l-m-\frac{1}{2})}{2^{2m+1}(m!)^2\Gamma(l+m+\frac{1}{2})}. \tag{C.30}$$

In addition to these equations, we will need to remember that $\Gamma(x+1) = x\Gamma(x)$ and

$$\Gamma\left(\frac{2n+1}{2}\right) = (2n-1)(2n-3)\cdots 3\cdot 1\frac{\sqrt{\pi}}{2^n}. \tag{C.31}$$

We will use these formulae in the section 6.5.2.

C.4 The Vanishing of the Cross Term in the CMB Power Spectrum

In this section we will show that the cross term of the monopole and dipole in the expression for the small scale C_l's (equation (6.48)) from section 6.5 vanishes for large l. The cross term is

$$C.T. \propto \int_0^\infty dx x^{3-n}\tilde{\Theta}_1\big[\tilde{\Theta}_0+\tilde{\Psi}\big]\big[J_{l+1/2}(x)J_{l-1/2}(x) - \frac{l+1}{x}J^2_{l+1/2}(x)\big]. \tag{C.32}$$

If we use the same approximation for the Bessel integrals as we did in section 6.5 (equations (6.51) and (6.62)), the only relevant x-dependence comes from the gravitational potential[1], which is of the order $\sim x^{-p}$ for positive integer

[1]This x dependence comes from the Transfer functions. Observe that we are also assuming a spectral index of $n = 1$.

p. Hence we are left several terms proportional to the expression below.

$$\mathrm{I} \equiv \int_0^\infty dx x^{-k} \big[J_{l+1/2}(x) J_{l-1/2}(x) - \frac{l+1}{x} J^2_{l+1/2}(x) \big]. \tag{C.33}$$

We shall now calculate the above integral. Multiplying through with the factor x^{-k} gives

$$\mathrm{I} = \int_0^\infty dx \big[x^{-k} J_{l+1/2}(x) J_{l-1/2}(x) - (l+1) x^{-k-1} J^2_{l+1/2}(x) \big]. \tag{C.34}$$

We have two possibilities, either k is odd or even.

Case 1: Even k

Using the Bessel integrals from section C.3, we obtain for even $k = 2m$

$$\begin{aligned}
\mathrm{I} &= \int_0^\infty dx \big[x^{-2m} J_{l+1/2}(x) J_{l-1/2}(x) - (l+1) x^{-2m-1} J^2_{l+1/2}(x) \big] \\
&= \frac{(2m-1)!\Gamma(l-m+\frac{1}{2})}{2^{2m} m!(m-1)!\Gamma(l+m+\frac{1}{2})} - (l+1)\frac{(2m)!\Gamma(l-m+\frac{1}{2})}{2^{2m+1}(m!)^2\Gamma(l+m+\frac{3}{2})} \\
&= \frac{\Gamma(l-m+\frac{1}{2})}{\Gamma(l+m+\frac{1}{2})} \left[\frac{(2m-1)!}{2^{2m} m!(m-1)!} - \frac{(2m)!(l+1)}{2^{2m+1}(m!)^2(l+m+\frac{1}{2})} \right] \\
&= \frac{\Gamma(l-m+\frac{1}{2})}{\Gamma(l+m+\frac{1}{2})} \frac{1}{2^{2m} m!} \left[\frac{(2m-1)!}{(m-1)!} - \frac{(2m)!(l+1)}{2(m)!(l+m+\frac{1}{2})} \right] \\
&= \frac{\Gamma(l-m+\frac{1}{2})}{\Gamma(l+m+\frac{1}{2})} \frac{(2m)!}{2^{2m} m!} \left[\frac{1}{2(m)!} - \frac{l+1}{2(m)!(l+m+\frac{1}{2})} \right],
\end{aligned} \tag{C.35}$$

where we have used that $\Gamma(l+m+\frac{3}{2}) = (l+m+\frac{1}{2})\Gamma(l+m+\frac{1}{2})$. As l becomes large, the first fraction[2] of gamma functions is always ≤ 1, and the l dependent ratio in the second term in the brackets tends to 1 . We hence get

$$\begin{aligned}
\mathrm{I} &\simeq \frac{(2m)!}{2^{2m} m!} \left[\frac{1}{2(m)!} - \frac{1}{2(m)!} \right] \\
&= 0,
\end{aligned} \tag{C.36}$$

which is what we wanted to prove.

[2]Observe that this always the case even when we multiply by $l(l+1)$ since since k is 2 at the least.

Case 2: Odd k

For odd values of k we set $k = 2m + 1$. We obtain

$$\begin{aligned}
\mathrm{I} &= \int_0^\infty dx \big[x^{-2m-1} J_{l+1/2}(x) J_{l-1/2}(x) - (l+1) x^{-2m-2} J_{l+1/2}^2(x)\big] \\
&= \frac{(2m)!(l-m-1)!}{2^{2m+1}\Gamma(m+\frac{3}{2})\Gamma(m+\frac{1}{2})(l+m)!} - \frac{(l+1)(2m+1)!(l-m-1)!}{2^{2m+2}\Gamma^2(m+\frac{3}{2})(l+m+1)!} \\
&= \frac{(l-m-1)!}{2^{2m+1}(l+m)!}\left[\frac{(2m)!}{\Gamma(m+\frac{3}{2})\Gamma(m+\frac{1}{2})} - \frac{(2m+1)!(l+1)}{2\Gamma^2(m+\frac{3}{2})(l+m)}\right] \\
&= \frac{(l-m-1)!}{2^{2m+1}(l+m)!}\frac{1}{\Gamma(m+\frac{3}{2})\Gamma(m+\frac{1}{2})}\left[(2m)! - \frac{(2m+1)!(l+1)}{2(m+\frac{1}{2})(l+m)}\right]. \qquad \text{(C.37)}
\end{aligned}$$

As we let l get large, the factorial fraction is at the most 1 since m is at least one in our setting[3]. Hence for large l we get

$$\begin{aligned}
\mathrm{I} &\simeq \frac{1}{2^{2m+1}}\frac{1}{\Gamma(m+\frac{3}{2})\Gamma(m+\frac{1}{2})}\left[(2m)! - \frac{(2m+1)!}{(2m+1)}\right] \\
&= \frac{1}{2^{2m+1}}\frac{1}{\Gamma(m+\frac{3}{2})\Gamma(m+\frac{1}{2})}\left[(2m)! - (2m)!\right] \\
&= 0. \qquad \text{(C.38)}
\end{aligned}$$

We have thus proved that the cross term does not contribute to the small scale CMB power spectrum and it is therefore justified not to include it in our calculations in section 6.5. This proof confirms a result from [10] which is done numerically.

[3] It actually tends to 0 for all m larger than one, even when we include the factor of $l(l+1)$.

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