Empirical Error Estimates for the Linearized Navier-Stokes Equations

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Abstract
Many mixed finite elements have been proposed to approximate the solution of the Navier-Stokes equations for laminar, incompressible fluid flow. The theory for these elements is centered around the stationary Stokes problem, but it is far from Stokes problem theory to real-life applications of mixed elements in, e.g., the car industry. This paper extends theoretical convergence estimates to the linearized Navier-Stokes equations and discusses the non-trivial influence of the time-stepping parameter.

Mardal, Tai and Winther (2002) constructed a new mixed finite element, which was robust with respect to a physical parameter ϵ appearing in Darcy-Stokes porous media flow. This element was proved to be much more accurate than well-known and popular elements for the Stokes/Navier-Stokes problem, when ϵ was small. The parameter ϵ is related to the time-stepping parameter in time-dependent flow. Therefore the accuracy of the standard elements for the time-dependent Navier-Stokes equations can be questioned. However, in time-dependent problems the error decreases as the time-stepping parameter decreases. It may be that the spatial accuracy is sufficient to ensure the convergence of the scheme, but this is not obvious. Our aim with this paper is to investigate the influence of the time-stepping parameter on the stability of mixed finite elements for time-dependent flow, in the simplified case where the nonlinear convective term is omitted. We compare the accuracy of several elements, such as the Taylor-Hood, Mini, Crouzeix-Raviart, $P_2 - P_0$ elements, as well as the previously mentioned new element. The numerical experiments suggest that Taylor-Hood, Mini and the new element are more accurate (in some sense) than Crouzeix-Raviart and $P_2 - P_0$, for flow with low viscosity. The results from these experiments are relevant to full Navier-Stokes solvers, as many of these have the linearized Navier-Stokes problem as one component in the solution method.

Key words: Mixed finite elements, computational fluid dynamics, time-dependent flow

INTRODUCTION

In this paper we will discuss the accuracy of different mixed finite elements for a simplified version of the Navier-Stokes equations for an incompressible fluid flow. The incompressible Navier-Stokes equations read

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\frac{1}{\varrho}\nabla p + \mu \Delta \mathbf{u} + \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned}$$

where $\mathbf{u}$ and p are the unknown velocity and pressure, respectively. The body forces is represented by $\mathbf{f}$, while μ is the viscosity and ϱ is the density of the fluid. Additionally, we must assign suitable boundary and initial conditions.

The Navier-Stokes equations are now solved routinely by today's scientists and engineers, using a variety of software packages and numerical methods. An overview of finite element methods for the incompressible Navier-Stokes equations can be found in Langtangen, Mardal and Winther (2002). Unfortunately, for many problems the solution obtained with different packages and methods differ. A quote from Turek (1999) illustrates this fact:

> There is no software available which can provide a guaranteed lift and drag coefficient on a car-body with an error tolerance of less that 20%; often the sign of the lift cannot even be predicted. Hence, we stopped flow around objects and use simulation tools for interior flow problems only, for instance for modeling heating devices or acoustic behavior in car cabins. Here, we are content with a qualitatively good prediction!

Notice that this quote also applies to flow with Reynolds numbers as low as 20. In fact, in M. Schäfer and S. Turek (1996) they conducted a benchmark for flow around a cylinder with Reynolds number 20. They compared the simulation results from various implementation of discretization techniques and solution methods. The solutions computed by 17 research groups differed by 20% in the computation of the lift coefficient. This problem is supposed to be "simple".

The uncertainty related to the efficiency and accuracy of different numerical methods actually calls for software where many different methods and formulations can easily be compared. Such flexibility was an important motivation behind the design of the generic C++ library Diffpack, Langtangen (2003). Diffpack has been extended with many mixed finite elements, see Mardal and Langtangen (2003), and offers the possibility to experiment with different mixed elements for fluid flow. This is exactly the purpose of the present paper; we want to conduct numerical experiments to learn more about the influence of the time-stepping parameter on the stability of mixed finite elements for incompressible viscous fluid flow. To the authors knowledge, Diffpack is the only software package that offers a range of mixed finite elements, for widely different applications, with ease of implementation also for unstructured grids in two and three space dimensions.

The mixed finite element methods that scientists and engineers are using routinely today to solve computational incompressible fluid dynamics problems have a seemingly solid mathematical basis. However, stability and general suitability of these elements are only proved for the stationary Stokes problem. That is, the local and convective acceleration terms are left out of the analysis. The next two sections outlines the potential instability of mixed finite elements that may occur in time-dependent flow problems as the time-stepping parameter tends to zero. Thereafter, we try to investigate how the theoretical results extend to time-dependent problems. Our method of investigation is based on conducting numerical experiments. We test various mixed finite elements applied to a time-dependent, but linearized, Navier-Stokes problem with known analytical solution such that we can measure errors and convergence rates exactly.

LINEARIZED NAVIER-STOKES EQUATIONS

The mathematical model to be addressed in this paper reads

$$\frac{\partial \mathbf{u}}{\partial t} - \mu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0\,. \tag{2}$$

We will refer to these equations as the time-dependent Stokes problem. The reason for studying this model, and not the full incompressible Navier-Stokes equations are two-fold. First, we want to investigate how stability and convergence estimate established mathematically for Stokes flow extend to more realistic cases. To this end, it is natural to first add time-dependency and then, as a next step, add the nonlinear convection term. Second, the model problem (1)–(2) constitutes an important component in many Navier-Stokes solvers. For example, a time-stepping strategy is to split the full nonlinear Navier-Stokes equations into simpler set of equations, where the time-dependent Stokes problem (i.e., linearized Navier-Stokes equations) appears as one building block in the numerical algorithm. Fractional step strategies, see for example Dean and Glowinski (1993), are often founded on this idea. Using Lagrangian coordinates "to remove" the convective term, cf. Pironneau (1989), is another solution strategy for the Navier-Stokes equations that leads to the requirement of efficient solvers for (1)–(2).

Discretizing by backward Euler in time we get the following sequence of linear systems to be solved,

$$\begin{aligned} \mathbf{u}^n - \mu\Delta t\Delta\mathbf{u} + k\nabla p &= \Delta t\mathbf{f} + \mathbf{u}^{n-1}, \\ \nabla\cdot\mathbf{u}^n &= = 0\,. \end{aligned}$$

Here Δt is the size of the time step, i.e., the time-stepping parameter, and n denotes the time level. Babuska-Brezzi stability conditions, uniform in Δt, can be found in Mardal and Winther (2003). A crucial point in the theory of error estimates is the ellipticity of the operator, which involves the parameter in front of the $-\Delta$-operator, i.e., $\mu\Delta t$. In the next section we briefly review the results from Mardal, Tai and Winther (2002), which discuss the accuracy of different elements in terms of the ellipticity parameter, though for a stationary version of our model problem (1)–(2).

RESULTS FROM A SIMPLIFIED MODEL PROBLEM

Mardal, Tai and Winther (2002) found that different mixed finite elements showed remarkably different behavior if μ was varied in the following problem:

$$\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \tag{3}$$

$$\nabla\cdot\mathbf{u} = 0\,. \tag{4}$$

The relevance of this problem for incompressible fluid flow is obvious: provided that the numerical method requires solutions of "time-dependent Stokes problems", and that these are discretized by a backward scheme, we arrive at (3)–(4), though with μ replaced by $\mu\Delta t$.

The accuracy of standard Stokes elements for (3)–(4) was found by Mardal et al. (2002) to decrease as μ decreases. This is a kind of instability of the elements as $\mu \to 0$. However, a new element, robust in μ, was constructed in that work. It was also found that elements with continuous pressure, such as the Mini element or the Taylor-Hood elements, had remarkably better convergence in terms of μ than elements with discontinuous pressure. To illustrate the differences, we have included Figure 1 from Mardal et al. (2002), which shows the convergence behavior for various elements. In this case μ was small: $2^{-16} \sim 6\cdot 10^{-6}$. A brief description of the elements are given in the appendix.

The left figure displays the error in velocity, whereas the right figure shows the error in the energy norm,

$$\|\mathbf{u}\|_\epsilon = (\|\mathbf{u}\|^2 + \|\nabla \cdot \mathbf{u}\|^2 + \epsilon^2 \|\nabla \mathbf{u}\|^2)^{1/2}, \tag{5}$$

where $\| \cdot \|$ denotes the usual L_2 norm. The energy norm is weaker than the norm we would use to express the error estimates for the stress. Notice that all of the methods converge asymptotically, but that not all are in the asymptotic regime. In practice, it is hard to determine whether we are in this regime or not. However, the Mini, the Taylor-Hood and the new element are in some sense always in the the asymptotic regime. A short description of the elements use in this paper is provided in the appendix.

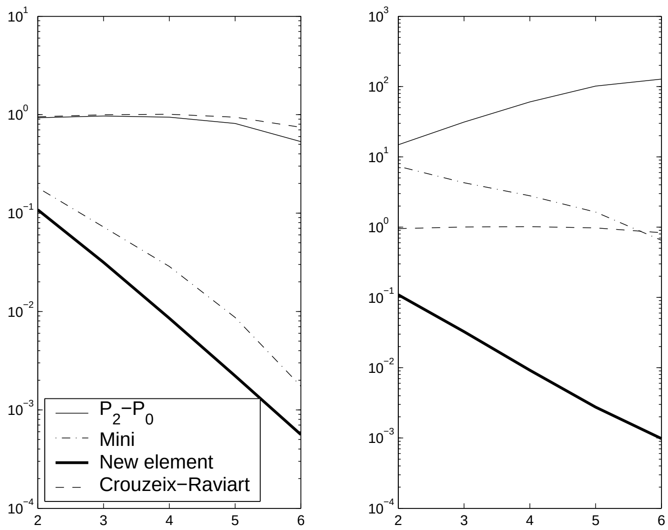

Fig. 1: The errors in velocity, measured in the L^2 norm (left) and the energy norm (right), as functions of the spatial discretization measure $\sigma = -\log(h)/\log(2)$ (h is the element size). The value of μ was $\sim 6 \cdot 10^{-6}$.

NUMERICAL EXPERIMENTS

Although it is clear that the different mixed finite element methods have different behavior in terms of μ in the problem cited in the previous section, it is not clear that this actually will affect the time-dependent problem (1)–(2). The important parameter in the time-dependent problem is $\Delta t \mu$, and it may be that the improvement in the Δt-error as $\Delta t \to 0$ is sufficient to handle the decrease in the spatial (element) error as $\mu \Delta t \to 0$. In other words, if we assume that the error in terms of h and Δt is,

$$e(h, \Delta t) = ch^\alpha + d(\Delta t)^\beta \, . \tag{6}$$

The point from Mardal et. al. (2002) is that for most elements c depend on Δt. This issue can be investigated through numerical experiments. We then need a flow problem with known exact solution such that we can compute the discretization error in each experiment. To this end, we manufacture a smooth solution, independent of μ,

$$\begin{aligned} u_1 &= -y \, sin(xy) e^{-t}, \\ u_2 &= x \, sin(xy) e^{-t}, \\ \mathbf{u} &= (u_1, u_2)^T, \\ p &= cos(xy) e^{-t} \, . \end{aligned}$$

This is a solution of our model problem (1)–(2) if we adjust the source term $\mathbf{f}$ to be

$$\mathbf{f} = \frac{\partial \mathbf{u}}{\partial t} - \mu\Delta\mathbf{u} + \nabla p\,.$$

The results of our experiments are listed in tables showing how the error in velocity varies with element size h and the time step Δt.

The results for $\mu = 1$ and the Crouzeix-Raviart element are shown in Table 1. The new robust element from Mardal et al. (2002) is shown in Table 2. We see that both elements have about the same accuracy. In the right-most column (where the spatial error is small) we see linear convergence in terms of Δt and in the lower-most row (where the error in time is small) we have roughly quadratic convergence in terms of h. We also have diagonal "stairs" with quadratic convergence. The same conclusion also apply to the Mini and $P_2 - P_0$ elements. The approximation with Taylor–Hood is of higher order in space, and the spatial convergence is barely visible. However, in all these numerical experiments, we conclude that the decrease in spatial accuracy is compensated by the increased accuracy in time.

$\Delta t \backslash h$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
1.00	**2.99e-2**	1.01e-2	5.24e-3	4.44e-3	**4.31e-3**
5.00e-1	3.14e-2	9.44e-3	3.28e-3	2.05e-3	**1.88e-3**
2.50e-1	3.20e-2	**9.33e-3**	2.72e-3	1.14e-3	**8.87e-4**
1.25e-1	3.22e-2	9.33e-3	2.56e-3	8.08e-4	**4.61e-4**
6.25e-2	3.23e-2	9.34e-3	**2.51e-3**	6.95e-4	**2.76e-4**
3.13e-2	**3.24e-2**	**9.34e-3**	**2.49e-3**	**6.57e-4**	**2.02e-4**

Table 1: The L_2 error of the velocity obtained with the Crouzeix-Raviart element for $\mu = 1$.

$\Delta t \backslash h$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
1.00	**1.50e-2**	5.33e-3	3.37e-3	3.03e-3	**2.95e-3**
5.00e-1	1.44e-2	4.20e-3	1.83e-3	1.37e-3	**1.29e-3**
2.50e-1	1.41e-2	**3.80e-3**	1.25e-3	7.05e-4	**6.05e-4**
1.25e-1	1.39e-2	3.64e-3	1.04e-3	4.25e-4	**3.05e-4**
6.25e-2	1.39e-2	3.58e-3	**9.52e-4**	3.07e-4	**1.68e-4**
3.13e-2	**1.39e-2**	**3.55e-3**	**9.15e-4**	**2.59e-4**	**1.04e-4**

Table 2: The L_2 error of the velocity obtained with the robust element for $\mu = 1$.

In the next example we run similar tests with a smaller viscosity parameter, $\mu = 10^{-4}$. The results for the Crouzeix-Raviart elements are shown in Table 3, and the results for the new robust element appear in Table 4. In this case we see clear differences in the accuracy, the robust element is clearly favorable. The Crouzeix-Raviart elements in not yet in the asymptotic range, because we see no sign of convergence either in the lower-most row or the right-most column. We also remark that the $P_2 - P_0$ elements behave similar to the Crouzeix-Raviart element in this case. On the other hand the robust element is convergent. We observe the same rates as in the case with $\mu = 1$.

$\Delta t\backslash h$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
1.00	**1.06e+0**	1.18e+0	1.09e+0	7.69e-1	**4.49e-1**
5.00e-1	5.06e-1	5.65e-1	5.55e-1	4.48e-1	**2.77e-1**
2.50e-1	2.17e-1	**2.36e-1**	2.48e-1	2.50e-1	**1.86e-1**
1.25e-1	1.12e-1	1.09e-1	1.02e-1	1.40e-1	**1.48e-1**
6.25e-1	1.21e-1	1.22e-1	**8.38e-2**	8.47e-2	**1.34e-1**
3.13e-1	**1.44e-1**	**1.52e-1**	**1.07e-1**	**6.07e-2**	**1.30e-1**

Table 3: The L_2 error of the velocity obtained with the Crouzeix-Raviart element for $\mu = 0.0001$.

$\Delta t\backslash h$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}
1.00	**2.01e-1**	2.11e-1	2.15e-1	2.18e-1	**2.19e-1**
5.00e-1	1.09e-1	1.16e-1	1.19e-1	1.20e-1	**1.21e-1**
2.50e-1	5.66e-2	**6.08e-2**	6.22e-2	6.30e-2	**6.34e-2**
1.25e-1	3.01e-2	3.09e-2	3.18e-2	3.22e-2	**3.24e-2**
6.25e-2	1.95e-2	1.55e-2	**1.60e-2**	1.63e-2	**1.64e-2**
3.13e-2	**1.69e-2**	**7.90e-3**	**8.01e-3**	**8.18e-3**	**8.25e-3**

Table 4: The L_2 error of the velocity obtained with the robust element for $\mu = 0.0001$.

The Mini and Taylor-Hood elements also show an accuracy comparable with the robust element.

CONCLUSION

In this paper we have seen that for relatively viscous fluids the error estimates for the Stokes problems are maintained also for the time-dependent linearized Navier-Stokes equations, implying that widely used mixed finite elements are well behaved. However, in fluids with low viscosities there are clear differences between the various mixed elements. The Mini, Taylor–Hood and the robust elements are preferable to the Crouzeix-Raviart and the $P_2 - P_0$ elements.

REFERENCES

E. J. Dean and R. Glowinski (1993). On Some Finite Element Methods for the Numerical Simulation of Incompressible Viscous Flow. *In Proceedings: Incompressible computational fluid dynamics; Trends and advances*, Editors: M. D. Gunzburger and R. A. Nicolaides, Cambridge University Press.

H. P. Langtangen (2003). Computational Partial Differential Equations – Numerical Methods and Diffpack Programming *Textbook in Computational Science and Engineering*, Springer.

H. P. Langtangen, K.-A. Mardal and R. Winther (2002). Numerical Methods for Incompressible Viscous Flow. Selected review paper for the 25th Anniversary Issue of the journal *Advances in Water Resources*, vol 25, 1125-1146.

K.- A. Mardal and H. P. Langtangen (2003). Mixed Finite Elements. *Advanced Topics in Computational Partial Differential Equation s – Numerical Methods and Diffpack Programming*. Springer.

K.- A. Mardal, X.-C. Tai and R. Winther. A Robust Finite Element Method for Darcy-Stokes Flow. *SIAM Journal on Numerical Analysis*. Vol 40. Pages 1605-1631.

K.-A. Mardal and R. Winther (2003). Uniform Preconditioners for the Time Dependent Stokes Problem, Submitted *Numerische Mathematik*

O. Pironneau (1989). The finite element method for fluids,
John Wiley & Sons.

M. Schäfer and S. Turek (1996). Benchmark computations of laminar flow around cylinder, *Flow Simulation with High-Performance Computers II*, E.H. Hirschel (ed), Vieweg.

S. Turek (1999). Efficient Solvers for Incompressible Flow Problems. *Lecture Notes in Computational Science and Engineering*. Springer.

APPENDIX: LIST OF ELEMENTS

We also provide a short description of the elements used in the experiments. The Crouzeix-Raviart element is shown in Figure 2. The black circles are velocity nodes, while the white circle shows the pressure node. This element is also often called the linear non-conforming element. The nodes are on the midpoints of each side and are the only place where the basis functions are continuous across element edges. This element should be combined with the piecewise constant pressure element, P_0.

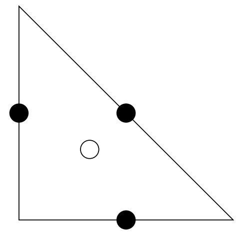

Fig. 2: Sketch of the 2D Crouzeix–Raviart element.

The Mini element is also popular. It consists of standard linear elements for the velocity, with an additional bubble function in the middle of the element. This element should be combined with linear continuous pressure elements. Figure 3 shows the Mini element. The black squares indicate that the node is associated both with the pressure and the velocity.

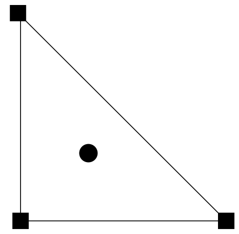

Fig. 3: Sketch of the 2D Mini velocity element and the linear pressure element.

The Taylor-Hood elements are piecewise quadratic and continuous velocity elements, which are combined with piecewise linear and continuous pressure elements. It is shown in Figure 4. The $P_2 - P_0$ element is similar to the Taylor-Hood element except that the pressure element is piecewise constant.

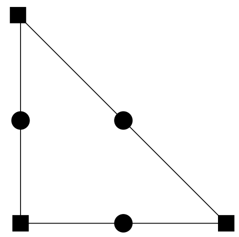

Fig. 4: Sketch of the 2D Taylor–Hood element; Quadratic velocity and linear pressure elements.

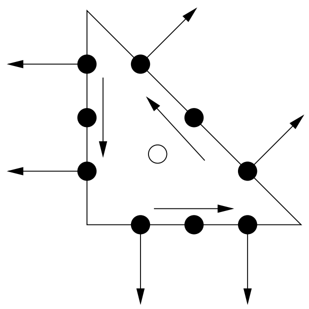

Fig. 5: The degrees of freedom of the robust element.

The new element from Mardal et. al. (2002) is shown in Figure 5. This is a slightly more complicated element, which is continuous only in the normal direction across element faces. However, the mean value of the tangential component is continuous. This element should be combined with piecewise constant pressure elements.