

# EFFICIENT PRECONDITIONERS FOR OPTIMALITY SYSTEMS ARISING IN CONNECTION WITH INVERSE PROBLEMS

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ABSTRACT. This paper is devoted to the numerical treatment of linear optimality systems (OS) arising in connection with inverse problems for partial differential equations. If such inverse problems are regularized by Tikhonov regularization, then it follows from standard theory that the associated OS is well-posed, provided that the regularization parameter  $\alpha$  is positive and that the involved state equation satisfies suitable assumptions.

We explain and analyze how certain mapping properties of the operators appearing in the OS can be employed to define efficient preconditioners for finite element (FE) approximations of such systems. The key feature of the scheme is that the number of iterations needed to solve the preconditioned problem by the minimal residual method is bounded independently of the mesh parameter  $h$ , used in the FE discretization, and only increases moderately as  $\alpha \rightarrow 0$ . More specifically, if the stopping criterion for the iteration process is defined in terms of the associated energy norm, then the number of iterations required (in the severely ill-posed case) cannot grow faster than  $O((\ln(\alpha))^2)$ . Our analysis is based on a careful study of the involved operators which yields the distribution of the eigenvalues of the preconditioned OS.

Finally, the theoretical results are illuminated by a number of numerical experiments addressing both a model problem studied by Borzi, Kunisch and Kwak [14] and an inverse problem arising in connection with electrocardiography [41].

## 1. INTRODUCTION

Let  $H_1$ ,  $H_2$  and  $H_3$  be Hilbert spaces with inner products  $(\cdot, \cdot)_{H_1}$ ,  $(\cdot, \cdot)_{H_2}$ ,  $(\cdot, \cdot)_{H_3}$ , norms  $\|\cdot\|_{H_1}$ ,  $\|\cdot\|_{H_2}$ ,  $\|\cdot\|_{H_3}$  and dual spaces  $H'_1$ ,  $H'_2$  and  $H'_3$ . We will consider parameter identification problems which can be written in the form

$$(1) \quad \min_{v \in H_1} \left\{ \frac{1}{2} \|Tu - d\|_{H_3}^2 + \frac{1}{2} \alpha \|v - v_{\text{prior}}\|_{H_1}^2 \right\}$$

subject to

$$(2) \quad Au = -Bv + g \quad (\text{state equation}),$$

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where

$$\begin{aligned} A &: H_2 \rightarrow H'_2, \\ B &: H_1 \rightarrow H'_2, \\ T &: H_2 \rightarrow H_3 \quad (\text{observation operator}) \end{aligned}$$

are bounded linear operators,  $d \in H_3$  and  $g \in H'_2$  are given quantities, and  $\alpha > 0$  is a regularization parameter. We are thus aiming at using an observation  $d \in H_3$  of  $Tu \in H_3$  to recover the parameter  $v \in H_1$  present in the state equation (2). Tikhonov regularization is applied, and  $v_{\text{prior}}$  represents a prior (invoking a priori knowledge) for  $v$ . In this paper we consider problems in which (2) is a partial differential equation (PDE). (Our preconditioning technique and analysis can be generalized in a rather straightforward manner to cases in which (2) is a linear system of PDEs or other well-behaved linear equations).

As is well-known, the solution of (1)-(2) must satisfy a saddle-point problem on the form

$$(3) \quad \begin{bmatrix} M_\alpha & N' \\ N & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}.$$

Here,  $x = (v, u)$ ,  $y$  is the Lagrange multiplier, and we will return to the exact structure of the operators  $M_\alpha$  and  $N$  below.

Preconditioners for saddle-point problems have been studied extensively the last decades [3, 10, 17, 19, 23, 26, 27, 32, 37, 42, 44, 47, 52]. Our work is based on the approach suggested in [3, 32, 37], where the saddle point problem is considered as an isomorphism between an appropriate Hilbert space and its dual space, provided that the Babuška-Brezzi conditions [20] are satisfied. By letting the preconditioner be an isomorphism mapping the dual space back to the Hilbert space, the preconditioned system becomes well-conditioned, and in the discrete case one obtains condition numbers which can be bounded independently of the mesh parameter  $h$ .

The problem studied in this paper is different from those analyzed in [3, 32, 37] because (1)-(2) typically is ill-posed for  $\alpha = 0$ . Furthermore, even though the regularized problem ( $\alpha > 0$ ) is well-posed, the Babuška-Brezzi conditions will involve constants that depend on  $\alpha$ . This dependency on the regularization parameter causes the condition number of the matrix associated with (3) to increase as the regularization parameter decreases towards zero. We will demonstrate that efficient solution methods can be constructed in spite of this dependency. More specifically, in the severely ill-posed case, it turns out that the number of iterations needed by the preconditioned minimal residual method can not grow any faster than of order  $O((\ln(\alpha))^2)$ , provided that the convergence is measured in the energy norm. Moreover, numerical experiments indicate that this theoretical worst case scenario bound is rather pessimistic.

Many researchers have studied numerical methods for PDE constrained optimization problems. Promising results have been reported for multigrid methods, see [1, 4, 11, 12, 13, 14, 15, 30, 46, 48] and references therein. Also

successful preconditioning schemes for a rather wide range of optimality systems are available [1, 8, 9, 29, 33, 38, 45]. Finally, strategies for accelerating iterative methods for inverse problems have been proposed and analyzed [24, 25].

The novelty of our approach is that we observe that certain preconditioners keep almost all the eigenvalues in intervals that can be bounded independently of both the mesh parameter  $h$  and the regularization parameter  $\alpha$ . It is well-known that Krylov solvers are very efficient when the spectrum is bounded except for a few isolated eigenvalues. This fact has been thoroughly explored for the Conjugate Gradient scheme by Axelsson and Lindskog [5, 6, 7]. We extend their results to the minimal residual method and hence prove that the combination of our preconditioner and Krylov solvers yields a very efficient approach.

In [1, 14, 33, 38, 45] results for iterative schemes that are independent of both  $h$  and  $\alpha$  are presented for various model problems (i.e. for special cases of elliptic and parabolic control problems). Furthermore, only a moderate increase in the workload as  $\alpha \rightarrow 0$  was observed in [48]. As mentioned above, for general PDE constrained optimization problems on the form (1)-(2), the analysis of our preconditioner yields convergence properties which are independent of  $h$  and of order  $O((\ln(\alpha))^2)$ . We obtain these results with rather mild assumptions on the state equation (2) and the observation operator  $T$ .

Our preconditioner is block diagonal where each block is a standard “off-the-shelves” elliptic preconditioner. Since the resulting preconditioner is symmetric and positive, the minimal residual method is used. Consequently, our scheme can be implemented in a rather straightforward manner, taking into use previously developed PDE software.

Many inverse problems can be written in the form (1)-(2); Section 3 contains two examples. First we consider the case analyzed in [14]. Thereafter an inverse problem arising in connection with electrocardiography is studied. In both examples we perform a series of numerical experiments demonstrating the numerical efficiency of our algorithm.

This text is organized as follows: Section 2 contains the necessary assumptions and the definition of our preconditioner. As mentioned above, in Section 3 two examples are presented. And, finally, Section 4 is devoted to theoretical considerations.

## 2. ASSUMPTIONS, OPTIMALITY SYSTEM AND BLOCK PRECONDITIONERS

In what follows, we assume that the forward (direct) mapping

$$(4) \quad F : H_1 \rightarrow H_3, \quad F = TA^{-1}B,$$

associated with (1)-(2), is not continuously invertible. That is, for  $\alpha = 0$ , (1)-(2) is ill-posed. Please note that  $H_1$  is the parameter space,  $H_2$  is the state space, and  $H_3$  is the observation space.

**2.1. Assumptions.** Throughout this text we assume that:

**A1:**  $A$  is a bounded linear operator.

**A2:** There exists a constant  $c_1 > 0$ , independent of the regularization parameter  $\alpha$ , such that

$$(5) \quad c_1 \|\phi\|_{H_2}^2 \leq \langle A\phi, \phi \rangle \quad \text{for all } \phi \in H_2.$$

**A3:**  $B$  is bounded and linear.

**A4:** The observation operator  $T$  is bounded and linear.

From **A1-A2** it follows that  $A$  is continuously invertible, and, combined with **A3**, this implies that the solution  $u$  of (2) depends continuously on  $v$  and  $g$ :

$$(6) \quad \|u\|_{H_2} \leq c_2 \left( \|v\|_{H_1} + \|g\|_{H'_2} \right) \quad \text{for all } v \in H_1 \text{ and all } g \in H'_2,$$

where  $c_2$  is a positive constant not depending on  $\alpha$ , i.e. (2) is well-posed.

Throughout this text we assume that the state equation (2) is a partial differential equation (PDE) or a finite element (FE) approximation of a PDE. In the latter case,  $H_1$ ,  $H_2$  and  $H_3$  are finite dimensional;

$$\dim(H_1) = \nu_1 < \infty,$$

$$\dim(H_2) = \nu_2 < \infty,$$

$$\dim(H_3) = \nu_3 < \infty,$$

and the operators  $A$ ,  $B$  and  $T$  depend on a mesh parameter  $h$ . However, if the FE discretization procedure is sound, then  $A$ ,  $B$  and  $T$  will inherit suitable bounds from their continuous counterparts. In such situations one can thus typically bound the quantities involved in **A1-A4** by  $h$  independent constants. We thus make the following assumption:

**A5:** There exist constants  $b_1$ ,  $b_2$  and  $b_3$ , which are independent of  $h$  and  $\alpha$ , such that

$$\|A\|_{L(H_2, H'_2)} \leq b_1,$$

$$\|B\|_{L(H_1, H'_2)} \leq b_2,$$

$$\|T\|_{L(H_2, H_3)} \leq b_3.$$

In addition we assume that the constants  $c_1$  and  $c_2$ , present in (5) and (6), do not depend on  $h$ .

It is important to distinguish between the FE operators and their associated matrices. The matrices, as mappings between Euclidean spaces, will in most cases have norms which cannot be bounded independently of  $h$ .

Recall the form (4) of the forward operator  $F$ . Since we have assumed that  $A$  is continuously invertible, the ill-posed nature of (1)-(2) must be inherited from  $T$  or/and  $B$ . The observation mapping  $T$  is typically a restriction to a boundary or some sort of imbedding. Such operations can usually not be continuously inverted and in most cases the equation; find  $h_2 \in H_2$  such that

$$Th_2 = h_3 \in H_3,$$

is ill-posed. In the theoretical part of this paper we will consider the severely ill-posed situation:

**A6:** In the finite dimensional case we assume that the eigenvalues  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{\nu_2}$  of  $T^*T : H_2 \rightarrow H_2$  satisfy

$$(7) \quad \sigma_i \leq b_4 e^{-b_5 i} \quad \text{for } i = 1, 2, \dots, \nu_2.$$

Here,  $b_4$  and  $b_5$  are positive constants not depending on  $\nu_2 = \dim(H_2) < \infty$ ,  $h$  or  $\alpha$ , and  $T^* : H_3 \rightarrow H_2$  denotes the adjoint of  $T$ . ( $T^*T$  is self-adjoint and positive semi-definite and therefore has  $\nu_2$  (including multiplicity) eigenvalues which are larger or equal to zero).

The description of Assumption  $\mathcal{A6}$  in the infinite dimensional setting is somewhat more involved and therefore omitted. Our analysis will, for the sake of clarity, be presented in the finite dimensional setting.

As long as  $\mathcal{A1}$ - $\mathcal{A6}$  are satisfied, it turns out that further assumptions about the operator  $B$  is not needed. We will return to this issue in Section 4.

For mildly ill-posed problems  $\mathcal{A6}$  is replaced with

$\mathcal{A7}$ : There exist positive constants  $b_6$  and  $\xi$  such that the eigenvalues  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{\nu_2}$  of  $T^*T$  satisfy

$$(8) \quad \sigma_i \leq b_6 i^{-\xi}, \quad \text{for } i = 1, 2, \dots, \nu_2,$$

where  $b_6$  and  $\xi$  do not depend on  $\nu_2 = \dim(H_2) < \infty$ ,  $h$  or  $\alpha$ .

A brief discussion of such cases are presented in Appendix B.

**2.2. Optimality system.** For the sake of completeness, let us now review how an all-at-once scheme for (1)-(2) can be derived. The associated Lagrangian  $L_\alpha$  reads

$$\begin{aligned} L_\alpha(v, u, w) &= \frac{1}{2} \|Tu - d\|_{H_3}^2 + \frac{1}{2} \alpha \|v - v_{\text{prior}}\|_{H_1}^2 \\ &\quad + \langle Au, w \rangle + \langle Bv, w \rangle - \langle g, w \rangle \end{aligned}$$

for  $v \in H_1$  and  $u, w \in H_2$ , with Fréchet derivatives

$$\begin{aligned} \left\langle \frac{\partial L_\alpha}{\partial v}, \phi \right\rangle &= \alpha(v - v_{\text{prior}}, \phi)_{H_1} + \langle B\phi, w \rangle \quad \text{for } \phi \in H_1, \\ \left\langle \frac{\partial L_\alpha}{\partial u}, \phi \right\rangle &= (Tu - d, T\phi)_{H_3} + \langle A\phi, w \rangle \quad \text{for } \phi \in H_2, \\ \left\langle \frac{\partial L_\alpha}{\partial w}, \phi \right\rangle &= \langle Au, \phi \rangle + \langle Bv, \phi \rangle - \langle g, \phi \rangle \quad \text{for } \phi \in H_2. \end{aligned}$$

From the first order necessary condition

$$\frac{\partial L_\alpha}{\partial v} = 0, \quad \frac{\partial L_\alpha}{\partial u} = 0, \quad \frac{\partial L_\alpha}{\partial w} = 0$$

we find that a minimizer of (1)-(2) must satisfy the optimality system

$$(9) \quad \alpha(v, \phi)_{H_1} + \langle B\phi, w \rangle = \alpha(v_{\text{prior}}, \phi)_{H_1} \quad \text{for all } \phi \in H_1,$$

$$(10) \quad (Tu, T\phi)_{H_3} + \langle A\phi, w \rangle = (d, T\phi)_{H_3} \quad \text{for all } \phi \in H_2,$$

$$(11) \quad \langle Au, \phi \rangle + \langle Bv, \phi \rangle = \langle g, \phi \rangle \quad \text{for all } \phi \in H_2.$$

In an all-at-once approach one seeks to solve (9)-(11) in a fully coupled manner, i.e. to solve the primal (11), the dual (10) and the optimality condition (9) simultaneously.

Please note that (9)-(11) can be written in the form: Find  $(v, u, w) \in H_1 \times H_2 \times H_2$  such that

$$(12) \quad \begin{bmatrix} \alpha L & 0 & B' \\ 0 & K & A' \\ B & A & 0 \end{bmatrix} \begin{bmatrix} v \\ u \\ w \end{bmatrix} = \begin{bmatrix} \alpha L v_{\text{prior}} \\ Qd \\ g \end{bmatrix},$$

where

$$\begin{aligned} A' : H_2 &\rightarrow H'_2, & w &\rightarrow \langle A\phi, w \rangle \forall \phi \in H_2, \\ B' : H_2 &\rightarrow H'_1, & w &\rightarrow \langle B\phi, w \rangle \forall \phi \in H_1, \end{aligned}$$

denote the dual operators of  $A$  and  $B$ , respectively, and

$$(13) \quad L : H_1 \rightarrow H'_1, \quad v \rightarrow (v, \phi)_{H_1} \quad \forall \phi \in H_1,$$

$$(14) \quad K : H_2 \rightarrow H'_2, \quad u \rightarrow (Tu, T\phi)_{H_3} = (T^*Tu, \phi)_{H_2} \quad \forall \phi \in H_2,$$

$$(15) \quad Q : H_3 \rightarrow H'_2, \quad d \rightarrow (d, T\phi)_{H_3} = (T^*d, \phi)_{H_2} \quad \forall \phi \in H_2.$$

For the sake of convenience, let us introduce the notation

$$(16) \quad \begin{aligned} \mathcal{A}_\alpha &= \begin{bmatrix} \alpha L & 0 & B' \\ 0 & K & A' \\ B & A & 0 \end{bmatrix}, \\ p &= \begin{bmatrix} v \\ u \\ w \end{bmatrix}, \\ b &= \begin{bmatrix} \alpha Lv_{\text{prior}} \\ Qd \\ g \end{bmatrix}. \end{aligned}$$

We thus get the compact form

$$(17) \quad \mathcal{A}_\alpha p = b$$

for (12).

**2.3. Preconditioning.** Even though the system (17) fits nicely into the classical framework for saddle point problems, provided that  $\alpha > 0$ , it is difficult to analyze our preconditioning scheme in terms of the standard norm on  $H_1 \times H_2 \times H_2$ . In fact, due to reasons that will become evident below, it turns out that it is convenient to employ the following,  $\alpha$  dependent, topology

$$(18) \quad X = H_1 \times H_2,$$

$$(19) \quad \begin{aligned} \|x\|_X^2 &= \|(x_1, x_2)\|_X^2 = \alpha \|x_1\|_{H_1}^2 + \alpha \|x_2\|_{H_2}^2 + \langle Kx_2, x_2 \rangle, \\ &= \alpha \|x_1\|_{H_1}^2 + \alpha \|x_2\|_{H_2}^2 + (T^*Tx_2, x_2)_{H_2}, \\ &= \alpha \|x_1\|_{H_1}^2 + \alpha \|x_2\|_{H_2}^2 + \|Tx_2\|_{H_3}^2 \quad \text{for } x = (x_1, x_2) \in X, \end{aligned}$$

$$(20) \quad Y = H_2,$$

$$(21) \quad \|y\|_Y^2 = \frac{1}{\alpha} \|y\|_{H_2}^2 \quad \text{for } y \in Y.$$

As will be discussed in detail in Section 4, assumptions  $\mathcal{A}1$ - $\mathcal{A}5$  and the Babuška-Brezzi conditions imply that

$$(22) \quad \mathcal{A}_\alpha : X \times Y \rightarrow (X \times Y)'$$

defines an isomorphism for every  $\alpha > 0$ .

Preconditioning techniques are usually defined and analyzed in terms of matrices which define operators between Euclidean spaces, see e.g. [5, 22].

However, as mentioned in the introduction, an alternative approach has been suggested in [3, 32, 37]: If

$$\mathcal{B}_\alpha : (X \times Y)' \rightarrow X \times Y$$

is an isomorphism, then

$$\mathcal{B}_\alpha \mathcal{A}_\alpha : X \times Y \rightarrow X \times Y$$

is well-behaved and we might, in the finite dimensional case, apply an iterative scheme to solve

$$(23) \quad \mathcal{B}_\alpha \mathcal{A}_\alpha p = \mathcal{B}_\alpha b,$$

cf. (17). The efficiency of such a scheme will of course depend on the spectral condition number of  $\kappa(\mathcal{B}_\alpha \mathcal{A}_\alpha)$  and the CPU costs associated with applying  $\mathcal{B}_\alpha \mathcal{A}_\alpha$  to an element  $p_n \in X \times Y$ .

If the FE method is applied, then  $\mathcal{A}_\alpha$  and  $\mathcal{B}_\alpha$  typically inherit the bounds which their continuous counterparts satisfy. More precisely,  $\|\mathcal{A}_\alpha\|_{L(X \times Y, (X \times Y)')}$ ,  $\|\mathcal{A}_\alpha^{-1}\|_{L((X \times Y)', X \times Y)}$ ,  $\|\mathcal{B}_\alpha\|_{L((X \times Y)', X \times Y)}$  and  $\|\mathcal{B}_\alpha^{-1}\|_{L(X \times Y, (X \times Y)')}$  are bounded independently of the mesh parameter  $h$  and, consequently,

$$\kappa(\mathcal{B}_\alpha \mathcal{A}_\alpha) = \|\mathcal{B}_\alpha \mathcal{A}_\alpha\|_{L(X \times Y, X \times Y)} \|(\mathcal{B}_\alpha \mathcal{A}_\alpha)^{-1}\|_{L(X \times Y, X \times Y)}$$

is well-behaved as  $h \rightarrow 0$ . The role of the regularization parameter  $\alpha$  is more complicated and will be discussed in detail in Section 4.

Based on these considerations, we propose to use a preconditioner of the form

$$(24) \quad \mathcal{B}_\alpha^{-1} = \begin{bmatrix} \alpha Q_1 & 0 & 0 \\ 0 & \alpha Q_2 + K & 0 \\ 0 & 0 & \frac{1}{\alpha} Q_2 \end{bmatrix},$$

where

$$Q_1 : H_1 \rightarrow H_1'$$

and

$$Q_2 : H_2 \rightarrow H_2'$$

are uniformly elliptic and bounded linear operators which are continuously invertible. That is, there exists a positive constant  $b_7$  such that for  $i = 1, 2$ :

$$(25) \quad b_7 \|\phi\|_{H_i}^2 \leq \langle Q_i \phi, \phi \rangle \quad \text{for all } \phi \in H_i.$$

In addition, we assume that

$$J_1^{-1} Q_1 : H_1 \rightarrow H_1 \quad \text{and} \quad J_2^{-1} Q_2 : H_2 \rightarrow H_2$$

are self-adjoint operators, where  $J_1 : H_1 \rightarrow H_1'$  and  $J_2 : H_2 \rightarrow H_2'$  are the Riesz maps.

Since  $K$  is positive semi-definite, see (14), condition (25) assures that we can use the minimal residual method to solve (23). The role of the scaling with respect to  $\alpha$  in (24) will become evident in Section 4.

The precise definition of  $Q_1$  and  $Q_2$  will of course depend on the application under consideration. Provided that  $J_2^{-1} A$  is self-adjoint, one could

typically use  $Q_1 = L$  and  $Q_2 = A$ , see (13) and assumptions  $\mathcal{A}1$ - $\mathcal{A}2$ , which yield

$$(26) \quad \mathcal{B}_\alpha^{-1} = \begin{bmatrix} \alpha L & 0 & 0 \\ 0 & \alpha A + K & 0 \\ 0 & 0 & \frac{1}{\alpha} A \end{bmatrix}.$$

However,  $\alpha L$ ,  $\alpha A + K$  or  $1/\alpha A$  may not be cheap to invert, and in practical situations we therefore might use multigrid preconditioners

$$(27) \quad \widehat{\mathcal{B}}_\alpha = \begin{bmatrix} \widehat{(\alpha L)^{-1}} & 0 & 0 \\ 0 & \widehat{(\alpha A + K)^{-1}} & 0 \\ 0 & 0 & \widehat{(\frac{1}{\alpha} A)^{-1}} \end{bmatrix}.$$

Here,  $\widehat{(\alpha L)^{-1}}$ ,  $\widehat{(\alpha A + K)^{-1}}$  and  $\widehat{(\frac{1}{\alpha} A)^{-1}}$  are (scalar) multigrid preconditioners. More precisely, in the examples presented in the next section,  $\widehat{(\alpha L)^{-1}}$ ,  $\widehat{(\alpha A + K)^{-1}}$  and  $\widehat{(\frac{1}{\alpha} A)^{-1}}$  are approximations of  $(\alpha L)^{-1}$ ,  $(\alpha A + K)^{-1}$  and  $(\frac{1}{\alpha} A)^{-1}$ , respectively, consisting of one V-cycle employing the symmetric Gauss-Seidel smoother [18, 31, 50].

Please note that our preconditioner is block diagonal where each block is defined in terms of a classical scalar preconditioner. This means that  $\widehat{\mathcal{B}}_\alpha$  can be implemented in a rather straightforward manner; in many cases old “scalar” software can be reused. Furthermore, since the involved operators are positive, we employ the minimal residual method to solve (23).

### 3. TWO EXAMPLES

**3.1. Example 1.** We will now consider the problem analyzed by Borzi, Kunisch and Kwak [14]:

$$(28) \quad \min_{v \in L^2(\Omega)} \left\{ \frac{1}{2} \|Tu - d\|_{L^2(\Omega)}^2 + \frac{1}{2} \alpha \|v\|_{L^2(\Omega)}^2 \right\}$$

subject to

$$(29) \quad -\Delta u = v + g \quad \text{in } \Omega,$$

$$(30) \quad u = 0 \quad \text{on } \partial\Omega,$$

where  $\Omega = (0, 1) \times (0, 1)$ ,  $\alpha > 0$  is a regularization parameter and  $d, g \in L^2(\Omega)$  are given functions.

In this case  $H_1 = L^2(\Omega)$ ,  $H_2 = H_0^1(\Omega)$  and  $H_3 = L^2(\Omega)$ . Furthermore, the observation operator  $T$  is simply the imbedding

$$T : H_0^1(\Omega) \hookrightarrow L^2(\Omega), \quad \phi \rightarrow \phi,$$



and

$$\begin{aligned} L : L^2(\Omega) &\rightarrow (L^2(\Omega))', & v &\rightarrow (v, \phi)_{L^2(\Omega)} & \forall \phi \in L^2(\Omega), \\ B : L^2(\Omega) &\rightarrow H^{-1}(\Omega), & v &\rightarrow -(v, T\phi)_{L^2(\Omega)} & \forall \phi \in H_0^1(\Omega), \\ A : H_0^1(\Omega) &\rightarrow H^{-1}(\Omega), & u &\rightarrow \int_{\Omega} \nabla u \cdot \nabla \phi \, dx & \forall \phi \in H_0^1(\Omega), \\ K : H_0^1(\Omega) &\rightarrow H^{-1}(\Omega), & u &\rightarrow (Tu, T\phi)_{L^2(\Omega)} & \forall \phi \in H_0^1(\Omega). \end{aligned}$$

From standard theory for elliptic PDEs and FE discretization of such equations it follows that assumptions  $\mathcal{A}1$ - $\mathcal{A}5$  are satisfied, provided that a suitable FE scheme is applied. A more thorough investigation is needed to explore the ill-posed properties of this problem. The next subsection addresses this issue.

3.1.1. *Ill-posed properties.* Figure 1 shows a  $(\ln(\sigma_i), \ln(i))$  plot of the eigenvalues of  $T^*T$ , sorted in decreasing order, computed on a grid with mesh parameter  $h = 2^{-6}$ . This graph indicates that there exist positive numbers  $b_6$  and  $\xi$  such that

$$\ln(\sigma_i) \approx \ln(b_6) - \xi \ln(i)$$

or

$$\sigma_i \approx b_6 i^{-\xi}.$$

That is, example (28)-(30) seems to satisfy Assumption  $\mathcal{A}7$ .

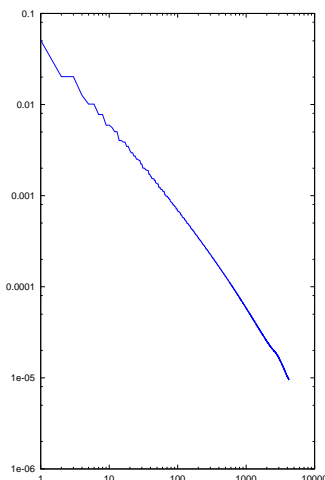


FIGURE 1. A  $(\ln(\sigma_i), \ln(i))$  plot of the eigenvalues of  $T^*T$  computed on a grid with mesh parameter  $h = 2^{-6}$  and sorted in decreasing order. Here,  $T : H_0^1(\Omega) \hookrightarrow L^2(\Omega)$  is the observation operator of the model problem discussed in Example 1.

To further explore the structure of this problem, let us equip  $H_2 = H_0^1(\Omega)$  with the inner product

$$\int_{\Omega} \nabla \psi \cdot \nabla \phi \, dx$$

and norm

$$\|\phi\|_{H_2}^2 = \int_{\Omega} |\nabla\phi|^2 dx$$

for  $\psi, \phi \in H_0^1(\Omega)$ .

Note that, for any  $v \in L^2(\Omega)$  and any  $\phi \in H_0^1(\Omega)$ ,

$$\begin{aligned} \langle Bv, \phi \rangle &= -(v, T\phi)_{L^2(\Omega)} \\ &= -(T^*v, \phi)_{H_0^1(\Omega)} \\ &= -\langle A(T^*v), \phi \rangle \end{aligned}$$

and we conclude that

$$B = -AT^*.$$

Consequently, we get the following formula for the forward operator

$$F = TA^{-1}B = -TA^{-1}AT^* = -TT^*,$$

and Figure 1 therefore also reveals the eigenvalue distribution of  $F$ . This graph thus indicates that (28)-(30) is mildly ill-posed.

3.1.2. *Numerical results.* Table 1 shows the numerical results obtained with the standard stopping criterion, which is defined in terms of the energy norm

$$\|\phi\|_{E,\alpha}^2 = \langle \mathcal{A}_\alpha\phi, \widehat{\mathcal{B}}_\alpha\mathcal{A}_\alpha\phi \rangle \quad \text{for } \phi \in H_1 \times H_2 \times H_2.$$

More specifically, the iteration process was stopped as soon as

$$(31) \quad \frac{\langle r_k, \widehat{\mathcal{B}}_\alpha r_k \rangle^{1/2}}{\langle r_0, \widehat{\mathcal{B}}_\alpha r_0 \rangle^{1/2}} = \frac{\|p_k - p^*\|_{E,\alpha}}{\|p_0 - p^*\|_{E,\alpha}} \leq 10^{-3},$$

where  $p^*$  and  $p_k$  represent the solution of (17) and the  $k$ th approximation of  $p^*$  generated by the preconditioned minimal residual method, respectively. The  $k$ th residual is denoted by  $r_k = \mathcal{A}_\alpha(p_k - p^*)$ . According to this table, the number of iterations needed to solve the problem seems to be bounded independently of the mesh parameter  $h$  and only increases moderately as  $\alpha$  decreases.

$h \setminus \alpha$	1	$10^{-1}$	$10^{-2}$	$10^{-3}$	$10^{-4}$
$2^{-1}$	4	4	4	4	4
$2^{-2}$	5	8	11	12	8
$2^{-3}$	7	8	12	17	14
$2^{-4}$	7	8	12	18	20
$2^{-5}$	9	10	12	17	21
$2^{-6}$	9	10	13	17	18
$2^{-7}$	8	10	13	15	16
$2^{-8}$	8	10	11	13	13
$2^{-9}$	8	8	9	11	12

TABLE 1. This table contains the number of iterations needed by the preconditioned minimal residual method to solve the model problem studied in Example 1. These numbers were generated with the standard stopping criterion (31) associated with the energy norm.

Let us consider the numbers presented in Table 1 in view of the classical estimate for the minimal residual method. This estimate states that

$$(32) \quad \frac{\|p_k - p^*\|_{E,\alpha}}{\|p_0 - p^*\|_{E,\alpha}} \leq 2 \left( \frac{\kappa(\widehat{\mathcal{B}}_\alpha \mathcal{A}_\alpha) - 1}{\kappa(\widehat{\mathcal{B}}_\alpha \mathcal{A}_\alpha) + 1} \right)^{\lceil k/2 \rceil - 1},$$

where  $\kappa(\widehat{\mathcal{B}}_\alpha \mathcal{A}_\alpha)$  is the spectral condition number of  $\widehat{\mathcal{B}}_\alpha \mathcal{A}_\alpha$ , see e.g. page 287 in [31]. (In Hackbusch's book the minimal residual method is referred to as the method of conjugate residuals). Here,  $\lceil a \rceil$ , for a real number  $a$ , denotes the smallest integer  $\geq a$ .

Inequality (32) indicates that the number of iterations needed by the minimal residual method will be of order  $O(\kappa(\widehat{\mathcal{B}}_\alpha \mathcal{A}_\alpha))$ . Table 2 contains  $\kappa(\mathcal{B}_\alpha \mathcal{A}_\alpha)$ , which clearly seems to be bounded independently of  $h$  but increases as the regularization parameter  $\alpha$  decreases. Consequently, (32) predicts accurately the performance, observed in Table 1, of the minimal residual method with respect to  $h$ , but provides a pessimistic estimate for the workload needed as  $\alpha \rightarrow 0$ .

$h \setminus \alpha$	1	$10^{-1}$	$10^{-2}$	$10^{-3}$	$10^{-4}$
$2^{-1}$	1.28	1.45	4.15	17.6	31.0
$2^{-2}$	1.34	1.61	5.07	16.9	52.3
$2^{-3}$	1.36	1.67	5.38	16.3	53.2
$2^{-4}$	1.37	1.68	5.46	16.2	53.5
$2^{-5}$	1.37	1.69	5.48	16.3	53.5

TABLE 2. This table contains numerical results obtained in Example 1. More precisely, the condition number  $\kappa(\mathcal{B}_\alpha \mathcal{A}_\alpha)$  of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$  for various grid refinement levels and  $\alpha = 1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$ .

To further shed some light onto this problem, we have plotted the absolute value of the eigenvalues of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$  in Figures 2 and 3. (In Figure 3 we have zoomed in on the smallest eigenvalues shown in Figure 2). Please note that almost all of the eigenvalues are of order  $O(1)$ . Krylov subspace solvers are known to handle such cases efficiently [5, 6, 7], and we will use this fact to explain the behavior, with respect to  $\alpha$ , observed in Table 1 in Section 4.

The energy norm depends on the regularization parameter  $\alpha$ . Hence, the stopping criterion (31) gets milder as  $\alpha$  decreases, which is also confirmed by the condition numbers reported in Table 2. We therefore decided to run a series of tests employing an  $\alpha$  independent stopping rule on the form

$$(33) \quad \frac{\langle \mathcal{A}_1(p_k - p^*), \widehat{\mathcal{B}}_1 \mathcal{A}_1(p_k - p^*) \rangle^{1/2}}{\langle \mathcal{A}_1(p_0 - p^*), \widehat{\mathcal{B}}_1 \mathcal{A}_1(p_0 - p^*) \rangle^{1/2}} \leq 10^{-3},$$

where  $\mathcal{A}_1 = \mathcal{A}_\alpha$  and  $\widehat{\mathcal{B}}_1 = \widehat{\mathcal{B}}_\alpha$  with  $\alpha = 1$ . Table 3 contains the iteration counts obtained in these experiments. The number of iterations required increases as  $\alpha$  decreases, but seems to be ‘‘rather bounded’’ independently of the mesh parameter  $h$ . The results are thus not as nice as those presented for the energy norm, compare tables 1 and 3.

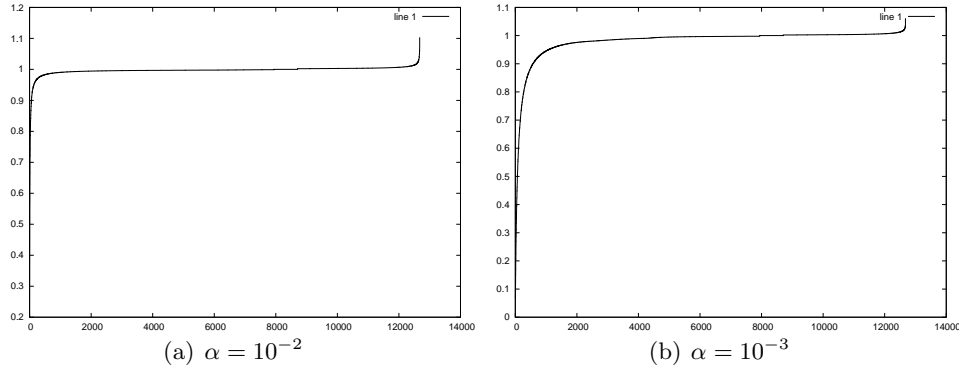


FIGURE 2. The absolute value of the eigenvalues of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$ , sorted in increasing order, in Example 1. These numbers were generated with mesh size  $h = 2^{-6}$  and regularization parameter  $\alpha = 10^{-2}, 10^{-3}$ .

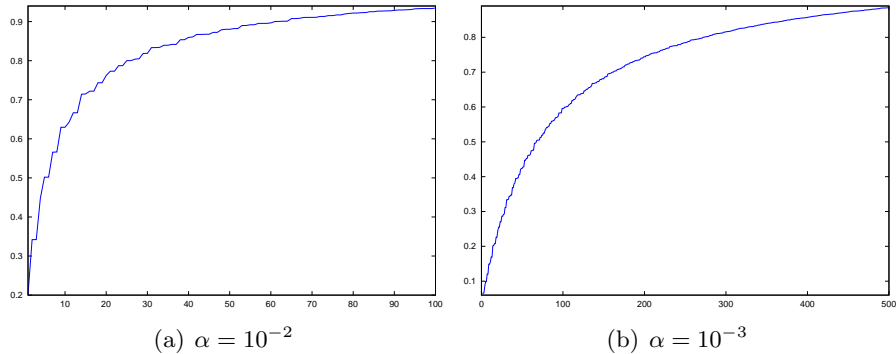


FIGURE 3. The hundred and five hundred smallest eigenvalues of those shown in Figure 2 (a) and Figure 2 (b), respectively.

**A remark.** In real world simulations, the exact solution  $p^*$  of the optimality system is not known. Consequently, the  $\alpha$  independent stopping criterion (33) cannot be used. The numbers presented in Table 3 were generated for a synthetic problem with  $p^* = 0$  and random initial guess  $p_0$  for the minimal residual method. On the other hand, since  $r_k = \mathcal{A}_\alpha(p_k - p^*) = \mathcal{A}_\alpha p_k - b$ , see (17), the classical rule (31) can always be employed - even if  $p^*$  is unknown.

**3.2. Example 2.** Our second example is the inverse transmembrane potential problem in electrocardiography. In this problem one seeks to use recordings of the electrical potential at the surface of the human body (ECG recordings) to compute the distribution of the so-called transmembrane potential  $v$  inside the heart. Many researchers have analyzed this important challenge, see e.g. [34, 39, 40, 41]. Usually, it is studied in terms of the bidomain model [36, 43, 49]. A detailed discussion of this issue is certainly

$h \setminus \alpha$	1	$10^{-1}$	$10^{-2}$	$10^{-3}$	$10^{-4}$
$2^{-1}$	4	4	5	5	5
$2^{-2}$	5	8	13	16	15
$2^{-3}$	7	8	15	26	27
$2^{-4}$	7	10	14	27	42
$2^{-5}$	9	10	14	28	49
$2^{-6}$	9	10	15	30	54
$2^{-7}$	8	10	14	32	54
$2^{-8}$	8	10	16	34	60
$2^{-9}$	8	10	17	35	64

TABLE 3. This table contains the number of iterations needed by the preconditioned minimal residual method to solve the model problem studied in Example 1. These numbers were generated with the  $\alpha$  independent stopping criterion (33).

beyond the scope of this text. We will therefore limit ourselves to presenting the mathematical formulation of the task and test the preconditioner proposed above on the associated optimality system.

Let  $H$  and  $G$  denote the domains occupied by the heart and torso, respectively, and define  $P = \overline{H} \cup G$ , with boundary  $\partial P$ , see Figure 4. In

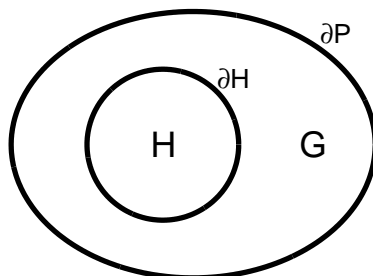


FIGURE 4. A schematic 2D illustration of the body  $P = \overline{H} \cup G$ , where  $H$  and  $G$  are the domains occupied by the heart and torso, respectively.

terms of mathematical symbols, we may express the inverse transmembrane potential problem on the form:

$$(34) \quad \min_{v \in H^1(H)} \left\{ \frac{1}{2} \|Tu - d\|_{L^2(\partial P)}^2 + \frac{1}{2} \alpha \|v - v_{\text{prior}}\|_{H^1(H)}^2 \right\}$$

subject to

$$(35) \quad \int_P (\mathbf{M} \nabla u) \cdot \nabla \phi \, dx = - \int_H (\mathbf{M}_i \nabla v) \cdot \nabla \phi \quad \text{for all } \phi \in H^1(P) \, dx,$$

where we, for the sake of convenience, have written the state equation in its variational form, see [41] for further information. Here,  $v$  and  $u$  are the

transmembrane and extracellular potentials, respectively, and  $\mathbf{M}$  and  $\mathbf{M}_i$  are conductivities. More specifically,

$$\mathbf{M}(x) = \begin{cases} \mathbf{M}_i(x) + \mathbf{M}_e(x) & \text{for } x \in \overline{H}, \\ \mathbf{M}_o(x) & \text{for } x \in G, \end{cases}$$

$\mathbf{M}_i$  is the intracellular conductivity of the heart,  $\mathbf{M}_e$  is the extracellular conductivity of the heart, and  $\mathbf{M}_o$  is the conductivity of the torso  $G$ .

We consider the two dimensional case, i.e.  $P \subset \mathbb{R}^2$  is a cross section of the body, and assume that all of the involved conductivities define uniformly positive definite tensors. That is, we assume that  $\mathbf{M}_o$ ,  $\mathbf{M}_i$  and  $\mathbf{M}_e$  are symmetric and that there exist positive constants  $\kappa_1$  and  $\kappa_2$  such that

$$(36) \quad 0 < \kappa_1 \leq \mathbf{M}_o(x) \leq \kappa_2 \quad \text{for all } x \in G,$$

$$(37) \quad 0 < \kappa_1 \leq \frac{\mathbf{a}^T \mathbf{M}_i(x) \mathbf{a}}{\mathbf{a}^T \mathbf{a}} \leq \kappa_2 \quad \text{for all } x \in \overline{H} \text{ and all } \mathbf{a} \in \mathbb{R}^2 \setminus \{0\},$$

$$(38) \quad 0 < \kappa_1 \leq \frac{\mathbf{a}^T \mathbf{M}_e(x) \mathbf{a}}{\mathbf{a}^T \mathbf{a}} \leq \kappa_2 \quad \text{for all } x \in \overline{H} \text{ and all } \mathbf{a} \in \mathbb{R}^2 \setminus \{0\}.$$

Now, referring to the symbols used in sections 1 and 2,  $H_1 = H^1(H)$ ,  $H_2 = H^1(P)$ ,  $H_3 = L^2(\partial P)$  and the observation operator  $T$  is simply the trace

$$(39) \quad T : H^1(P) \rightarrow L^2(\partial P), \quad \phi \rightarrow \phi|_{\partial P}.$$

Furthermore,

$$L : H^1(H) \rightarrow (H^1(H))', \quad \psi \rightarrow (\psi, \phi)_{H^1(H)} \quad \forall \phi \in H^1(H),$$

$$B : H^1(H) \rightarrow (H^1(P))', \quad \psi \rightarrow - \int_H (\mathbf{M}_i \nabla \psi) \cdot \nabla (R\phi) dx \quad \forall \phi \in H^1(P),$$

$$K : H^1(P) \rightarrow (H^1(P))', \quad \psi \rightarrow (T\psi, T\phi)_{L^2(\partial P)} \quad \forall \phi \in H^1(P),$$

$$A : H^1(P) \rightarrow (H^1(P))', \quad \psi \rightarrow \int_P (\mathbf{M} \nabla \psi) \cdot \nabla \phi dx \quad \forall \phi \in H^1(P),$$

where

$$R : H^1(P) \rightarrow H^1(H), \quad \phi \rightarrow \phi|_H$$

denotes the restriction operator.

**A remark.** For this example it is easy to verify that  $\mathcal{A}1$ ,  $\mathcal{A}3$  and  $\mathcal{A}4$  are satisfied, but that Assumption  $\mathcal{A}2$  does not hold. If  $u$  solves (35), so does  $u + c$  for any constant  $c$ . On the other hand, as is explained in [41], the solution  $(v, u) \in H^1(H) \times H^1(P)$  of the optimality problem (34)-(35) is unique, provided that  $\alpha > 0$ , and must satisfy

$$\int_{\partial P} u dx = \int_{\partial P} d dx.$$

Consequently, a formulation of (34)-(35) which yields a state equation with a unique solution is obtained as follows:

$$(40) \quad \min_{v \in H^1(H)} \left\{ \frac{1}{2} \left\| T\tilde{u} + \frac{1}{|\partial P|} \int_{\partial P} d dx - d \right\|_{L^2(\partial P)}^2 + \frac{1}{2} \alpha \|v - v_{\text{prior}}\|_{H^1(H)}^2 \right\}$$

subject to: Find  $\tilde{u} \in \tilde{H}^1(P)$  such that

$$(41) \quad \int_P (\mathbf{M}\nabla\tilde{u}) \cdot \nabla\phi \, dx = - \int_H (\mathbf{M}_i\nabla v) \cdot \nabla\phi \quad \text{for all } \phi \in \tilde{H}^1(P) \, dx,$$

where

$$(42) \quad \tilde{H}^1(P) = \left\{ \psi \in H^1(P); \int_{\partial P} T\psi \, dx = 0 \right\}.$$

More precisely,  $(v, \tilde{u}) \in H^1(H) \times \tilde{H}^1(P)$  solves (40)-(41) if and only if  $(v, u) = (v, \tilde{u} + \frac{1}{|\partial P|} \int_{\partial P} d \, dx) \in H^1(H) \times H^1(P)$  solves (34)-(35). A similar connection between the optimality systems associated with (40)-(41) and (34)-(35) can, of course, also be established. We will not dwell any further upon this issue.

**3.2.1. Ill-posed properties.** The shapes of the heart  $H$  and the body  $P$  are not simple. Consequently, we used nonuniform meshes in the FE discretization procedure. In all the figures and tables presented in connection with (34)-(35),  $l$  represents the refinement level of the grid. More precisely, as  $l$  increases the mesh size  $h$  decreases. ( $l$  is the number of times an initial coarse mesh has been refined).

Figure 5 shows a  $(\ln(\sigma_i), i)$  plot of the thirty largest eigenvalues of  $F^*F$  computed on mesh refinement level  $l = 0$ , where  $F$  is the forward mapping (4) associated with the model problem (34)-(35). The involved grid contained 1192 nodes of which 312 were located in the heart  $H$ . The remaining eigenvalues of  $F^*F$  were smaller than  $1.9292 * 10^{-16}$ .

Based on Figure 5, it seems reasonable to assume that there exist positive numbers  $b_4$  and  $b_5$  such that

$$\ln(\sigma_i) \approx \ln(b_4) - b_5 i$$

or

$$\sigma_i \approx b_4 e^{-b_5 i},$$

which indicates that the present problem is severely ill-posed.

In order to further explore the properties of Example 2, let us consider the formulation (40)-(41) of this problem. From the following modified version of Friedrichs' inequality

$$\int_P \phi^2 \, dx \leq C_1 \left( \int_P |\nabla\phi|^2 \, dx + \left\{ \int_{\partial P} \phi \, dx \right\}^2 \right) \quad \text{for all } \phi \in H^1(P),$$

see [35] and references therein, and assumptions (36)-(38) it follows that one can equip  $\tilde{H}^1(P)$ , defined in (42), with the inner product

$$(\psi, \phi)_{\tilde{H}^1(P)} = \int_P (\mathbf{M}\nabla\psi) \cdot \nabla\phi \, dx$$

and norm

$$\|\phi\|_{\tilde{H}^1(P)}^2 = (\phi, \phi)_{\tilde{H}^1(P)}$$

for  $\psi, \phi \in \tilde{H}^1(P)$ .

Consider the functions in  $\tilde{H}^1(P)$  that are zero on the heart, i.e.

$$(43) \quad V = \left\{ \psi \in \tilde{H}^1(P); \psi|_{\overline{H}} = 0 \right\}$$

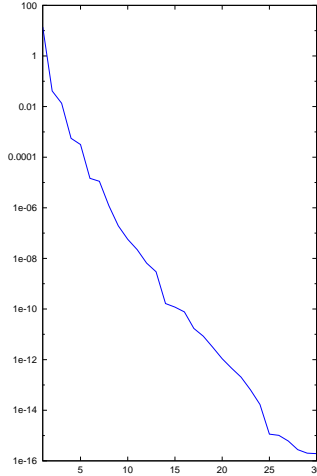


FIGURE 5. A  $(\ln(\sigma_i), i)$  plot of the thirty largest eigenvalues of  $F^*F$  computed on the coarsest grid, i.e.  $l = 0$ , and sorted in decreasing order. Here,  $F$  is the forward mapping associated with the model problem discussed in Example 2.

and its orthogonal complement

$$(44) \quad W = V^\perp = \left\{ \psi \in \tilde{H}^1(P); (\psi, \phi)_{\tilde{H}^1(P)} = 0 \text{ for all } \phi \in V \right\}.$$

Note that the solution  $\tilde{u}$  of (41) belongs to  $W$ , i.e.  $\tilde{A}^{-1}\tilde{B}v \in W$  for all  $v \in H^1(H)$ , where

$$\begin{aligned} \tilde{A} : \tilde{H}^1(P) &\rightarrow (\tilde{H}^1(P))', & \psi &\rightarrow \int_P (\mathbf{M}\nabla\psi) \cdot \nabla\phi \, dx \quad \forall \phi \in \tilde{H}^1(P), \\ \tilde{B} : H^1(H) &\rightarrow (\tilde{H}^1(P))', & \psi &\rightarrow - \int_H (\mathbf{M}_i\nabla\psi) \cdot \nabla(\tilde{R}\phi) \, dx \quad \forall \phi \in \tilde{H}^1(P), \\ \tilde{R} : \tilde{H}^1(P) &\rightarrow H^1(H), & \psi &\rightarrow \psi|_H. \end{aligned}$$

In fact, in Appendix A we prove that the range  $\mathcal{R}(\tilde{A}^{-1}\tilde{B})$  of  $\tilde{A}^{-1}\tilde{B}$  encompasses all of  $W$ , i.e.  $\mathcal{R}(\tilde{A}^{-1}\tilde{B}) = W$ , and that this mapping is a one-to-one operator. Since  $W$  is closed, it therefore follows from the Bounded Inverse Theorem that  $\tilde{A}^{-1}\tilde{B} : H^1(H) \rightarrow W$  is continuously invertible.

The ill-posed properties of the forward operator  $\tilde{F} = \tilde{T}\tilde{A}^{-1}\tilde{B}$  is hence (solely) inherited from the observation operator  $\tilde{T} = T|_W$ , where  $T$  is defined in (39). With other words, the decay of the singular values of  $\tilde{F}$  and  $\tilde{T}$  are closely related.

**3.2.2. Numerical results.** Also for this example the iteration counts obtained with the multigrid preconditioner  $\tilde{\mathcal{B}}_\alpha$  and the standard stopping criterion (31) are well-behaved, see Table 4. Indeed, the number of iterations needed seems to be bounded independently of both  $h$  and  $\alpha$ .

According to Table 5, the condition number  $\kappa(\mathcal{B}_\alpha\mathcal{A}_\alpha)$  of  $\mathcal{B}_\alpha\mathcal{A}_\alpha$  seems to be approximately of order  $O(\alpha^{-1})$ . Hence, the results reported in Table 4 are far better than what one would expect from the standard estimate (32). As observed in Example 1, almost all of the eigenvalues of  $\mathcal{B}_\alpha\mathcal{A}_\alpha$  are of order



$l \setminus \alpha$	1	$10^{-1}$	$10^{-2}$	$10^{-3}$	$10^{-4}$
0	32	40	55	42	25
1	28	36	49	52	24
2	26	30	41	51	26
3	28	28	36	47	32
4	29	28	32	41	41

TABLE 4. The number of preconditioned minimal residual iterations needed to solve the model problem studied in Example 2. These results were generated with the energy stopping criterion (31). Here,  $l$  is the refinement level of the grid, i.e. the mesh size  $h$  decreases as  $l$  increases.

$l \setminus \alpha$	1	$10^{-1}$	$10^{-2}$	$10^{-3}$	$10^{-4}$
1	16	108	672	5000	29729
2	16	109	680	5076	40157

TABLE 5. This table contains numerical results obtained in Example 2. More precisely, the condition number  $\kappa(\mathcal{B}_\alpha \mathcal{A}_\alpha)$  of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$  for various grid refinement levels  $l$  and  $\alpha = 1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$ . (The mesh size  $h$  decreases as  $l$  increases).

$O(1)$ , see Figure 6. We will return to this issue in the next section.

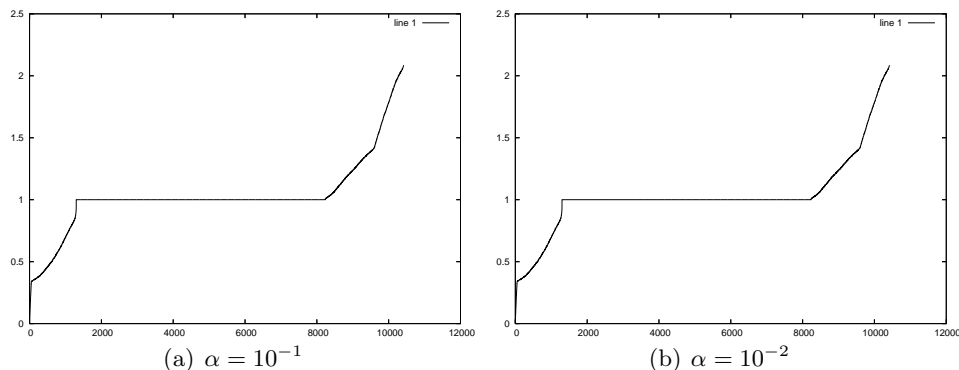


FIGURE 6. The absolute value of the eigenvalues of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$ , sorted in increasing order, computed on mesh level  $l = 2$  with regularization parameter  $\alpha = 10^{-1}, 10^{-2}$ . These results were obtained for the model problem studied in Example 2.

If the  $\alpha$  independent stopping rule (33) is used, then the workload increases as  $\alpha$  decreases, see Table 6. Nevertheless, the number of iterations needed does not “explode” for small values of  $\alpha$ . (The results presented in Table 6 were generated with  $p^* = 0$  and a random initial guess  $p_0$  for the minimal residual method).

$l \setminus \alpha$	1	$10^{-1}$	$10^{-2}$	$10^{-3}$
0	32	100	358	588
1	28	71	237	771
2	26	54	188	895
3	28	53	179	688
4	29	46	150	494

TABLE 6. The number of preconditioned minimal residual iterations needed to solve the model problem studied in Example 2. These results were generated with the  $\alpha$  independent stopping criterion (33). For  $\alpha \leq 10^{-4}$  instabilities occurred. Here,  $l$  is the refinement level of the grid, i.e. the mesh size  $h$  decreases as  $l$  increases.

#### 4. THEORETICAL CONSIDERATIONS

This section is devoted to a theoretical study of the preconditioning strategy proposed and tested above. We have seen that the standard estimate (32) cannot explain the results presented in tables 1 and 4. In order to analyze these observations, we will show that  $\mathcal{A}_\alpha$  is bounded, that the Babuška-Brezzi conditions hold and characterize the eigenvalues of the preconditioned operator  $\mathcal{B}_\alpha \mathcal{A}_\alpha$ . More specifically, it turns out that almost all of the eigenvalues are bounded independently of the mesh size  $h$  and the regularization parameter  $\alpha$ . This fact is combined with certain properties of Krylov subspace solvers to prove the main result of this paper; in the severely ill-posed case the number of iterations needed by the minimal residual method to solve (23) is bounded independently of  $h$  and cannot grow faster than of order  $O((\ln(\alpha))^2)$ .

For the sake of convenience, we consider the finite dimensional case in this section. That is,

$$(45) \quad \dim(H_1) = \nu_1 < \infty,$$

$$(46) \quad \dim(H_2) = \nu_2 < \infty,$$

$$(47) \quad \dim(H_3) = \nu_3 < \infty,$$

where  $\nu_1, \nu_2, \nu_3$  are positive integers. Please keep in mind that we assume that the operator norms of  $A, B$  and  $T$  are bounded independently of  $h$  and  $\alpha$ , see Assumption  $\mathcal{A}5$ , and that the constants  $c_1, c_2, b_4$  and  $b_5$  in (5), (6) and (7) do not depend on  $h$  or  $\alpha$ .

If we introduce the notation

$$(48) \quad M_\alpha = \begin{bmatrix} \alpha L & 0 \\ 0 & K \end{bmatrix} : X \rightarrow X',$$

$$(49) \quad N = \begin{bmatrix} B & A \end{bmatrix} : X \rightarrow Y',$$

$$f = \begin{bmatrix} \alpha L v_{\text{prior}} \\ Qd \end{bmatrix},$$

then we can write (12) in the form: Find  $x = (x_1, x_2) \in X$  and  $y \in Y$  such that

$$(50) \quad M_\alpha x + N'y = f,$$

$$(51) \quad Nx = g.$$

Please note that, for  $x = (x_1, x_2) \in X$ ,  $z = (z_1, z_2) \in X$ , and  $y \in Y$ ,

$$(52) \quad \begin{aligned} \langle M_\alpha x, z \rangle &= \alpha \langle x_1, z_1 \rangle_{H_1} + \langle Kx_2, z_2 \rangle \\ &= \alpha \langle x_1, z_1 \rangle_{H_1} + \langle Tx_2, Tz_2 \rangle_{H_3}, \end{aligned}$$

$$(53) \quad \langle Nx, y \rangle = \langle Bx_1, y \rangle + \langle Ax_2, y \rangle,$$

and recall the definitions (19) and (21) of the  $\|\cdot\|_X$  and  $\|\cdot\|_Y$  norms.

**4.1. Continuity.** We will now show that the operators  $M_\alpha$  and  $N$  are continuous and that the involved constants do not depend on the regularization parameter  $\alpha$  or on the mesh size  $h$ .

**Lemma 4.1.** *The mappings  $M_\alpha$  and  $N$ , defined in (48) and (49), satisfy*

- a)  $|\langle M_\alpha x, z \rangle| \leq 2 \|x\|_X \|z\|_X$  for all  $x, z \in X$ ,
- b)  $|\langle Nx, y \rangle| \leq (\|B\| + \|A\|) \|x\|_X \|y\|_Y$   
 $\leq (b_1 + b_2) \|x\|_X \|y\|_Y$  for all  $x \in X$  and all  $y \in Y$ ,

where  $\|B\| = \|B\|_{L(H_1, H_2')}$  and  $\|A\| = \|A\|_{L(H_2, H_2')}$  denote the operator norms of  $B$  and  $A$ , and  $b_1$  and  $b_2$  are independent of  $h$  and  $\alpha$  (see Assumption A5).

*Proof.* Let  $x, z \in X$  be arbitrary. The triangle and Cauchy-Schwarz inequalities imply that

$$\begin{aligned} |\langle M_\alpha x, z \rangle| &\leq \alpha |\langle x_1, z_1 \rangle_{H_1}| + |\langle Tx_2, Tz_2 \rangle_{H_3}| \\ &\leq \sqrt{\alpha} \|x_1\|_{H_1} \sqrt{\alpha} \|z_1\|_{H_1} + \|Tx_2\|_{H_3} \|Tz_2\|_{H_3} \\ &\leq \|x\|_X \|z\|_X + \|x\|_X \|z\|_X \\ &= 2 \|x\|_X \|z\|_X, \end{aligned}$$

which finishes the proof of part a).

Part b) is a consequence of the assumed boundedness of the operators  $A$  and  $B$  present in the state equation (2);

$$\begin{aligned} |\langle Nx, y \rangle| &\leq |\langle Bx_1, y \rangle| + |\langle Ax_2, y \rangle| \\ &\leq \|B\| \|x_1\|_{H_1} \|y\|_{H_2} + \|A\| \|x_2\|_{H_2} \|y\|_{H_2} \\ &= \|B\| \sqrt{\alpha} \|x_1\|_{H_1} \frac{1}{\sqrt{\alpha}} \|y\|_{H_2} + \|A\| \sqrt{\alpha} \|x_2\|_{H_2} \frac{1}{\sqrt{\alpha}} \|y\|_{H_2} \\ &\leq \|B\| \|x\|_X \|y\|_Y + \|A\| \|x\|_X \|y\|_Y \\ &= (\|B\| + \|A\|) \|x\|_X \|y\|_Y. \end{aligned}$$

□

From this lemma it follows that  $\mathcal{A}_\alpha : X \times Y \rightarrow (X \times Y)'$  is bounded (uniformly with respect to  $\alpha$  and  $h$ ).

**4.2. Babuška-Brezzi conditions.** The purpose of this section is to show that  $M_\alpha$  is coercive on the kernel of  $N$  and that the famous inf-sup condition for saddle point problems is satisfied. It turns out that the coercivity condition holds independently of the regularization parameter  $\alpha$  and that the inf-sup condition involves a constant of order  $O(\sqrt{\alpha})$ . The details are as follows:

**Lemma 4.2.** *There exists a constant  $c_3$ , independent of  $\alpha$  and  $h$ , such that*

$$\langle M_\alpha z, z \rangle \geq c_3 \|z\|_X^2 \quad \text{for all } z \in Z = \{z \in X; Nz = 0\},$$

where  $M_\alpha$  is defined in (48).

*Proof.* This inequality is a consequence of the assumed well-posedness of the state equation (2). More precisely, let  $z = (z_1, z_2) \in X$  be such that  $Nz = 0$ , i.e.

$$Az_2 = -Bz_1.$$

Then (6) implies that

$$(54) \quad \|z_2\|_{H_2} \leq c_2 \|z_1\|_{H_1},$$

and from (52) and (54) we find that

$$\begin{aligned} \langle M_\alpha z, z \rangle &= \alpha \|z_1\|_{H_1}^2 + \|Tz_2\|_{H_3}^2 \\ &\geq \frac{\alpha}{2} \|z_1\|_{H_1}^2 + \frac{\alpha}{2c_2^2} \|z_2\|_{H_2}^2 + \|Tz_2\|_{H_3}^2 \\ &\geq c_3 (\alpha \|z_1\|_{H_1}^2 + \alpha \|z_2\|_{H_2}^2 + \|Tz_2\|_{H_3}^2) \\ &= c_3 \|z\|_X^2, \end{aligned}$$

where

$$c_3 = \frac{1}{2} \min \left( 1, \frac{1}{c_2^2} \right).$$

□

**Lemma 4.3.** *There exists a constant  $c_4 > 0$  such that the operator  $N$ , see (49), satisfies*

$$\inf_{y \in Y} \sup_{x \in X} \frac{\langle Nx, y \rangle}{\|x\|_X \|y\|_Y} \geq c_4 \sqrt{\alpha},$$

provided that  $0 < \alpha \leq 1$ .

*Proof.* Let  $y \in Y$  be arbitrary and define

$$\hat{x} = (0, y) \in X,$$

see (18) and (20). Clearly,

$$\sup_{x \in X} \frac{\langle Nx, y \rangle}{(\|x\|_X \|y\|_Y)} \geq \frac{\langle N\hat{x}, y \rangle}{(\|\hat{x}\|_X \|y\|_Y)}$$

and

$$\langle N\hat{x}, y \rangle = \langle B0, y \rangle + \langle Ay, y \rangle = \langle Ay, y \rangle \geq c_1 \|y\|_{H_2}^2,$$

see Assumption  $\mathcal{A}2$  (inequality (5)). Furthermore, for  $\alpha \in (0, 1]$ ,

$$\begin{aligned} \|\hat{x}\|_X \|y\|_Y &= \|(0, y)\|_X \|y\|_Y \\ &= [\alpha \|y\|_{H_2}^2 + \|Ty\|_{H_3}^2]^{\frac{1}{2}} \left[ \frac{1}{\alpha} \|y\|_{H_2}^2 \right]^{\frac{1}{2}} \\ &\leq [\alpha \|y\|_{H_2}^2 + \|T\|^2 \|y\|_{H_2}^2]^{\frac{1}{2}} \left[ \frac{1}{\alpha} \|y\|_{H_2}^2 \right]^{\frac{1}{2}} \\ &\leq \sqrt{1 + \|T\|^2} \|y\|_{H_2} \frac{1}{\sqrt{\alpha}}. \end{aligned}$$

We can thus conclude that

$$\begin{aligned} \frac{\langle N\hat{x}, y \rangle}{\|\hat{x}\|_X \|y\|_Y} &\geq \frac{c_1 \|y\|_{H_2}^2}{\sqrt{1 + \|T\|^2} \|y\|_{H_2} \frac{1}{\sqrt{\alpha}}} \\ &= \frac{c_1}{\sqrt{1 + \|T\|^2}} \sqrt{\alpha}. \end{aligned}$$

Since  $y \in Y$  was arbitrary, it follows that

$$\inf_{y \in Y} \sup_{x \in X} \frac{\langle Nx, y \rangle}{\|x\|_X \|y\|_Y} \geq c_4 \sqrt{\alpha},$$

where, see Assumption  $\mathcal{A}5$ ,

$$c_4 = \frac{c_1}{\sqrt{1 + b_3^2}}.$$

□

From lemmas 4.1, 4.2 and 4.3 it follows that

$$(55) \quad \mathcal{A}_\alpha = \begin{bmatrix} M_\alpha & N' \\ N & 0 \end{bmatrix}$$

defines an isomorphism between  $X \times Y$  and its dual  $X' \times Y'$ , see e.g. [16]. Furthermore, these standard results also provide upper bounds for the operator norms of  $\mathcal{A}_\alpha$  and  $\mathcal{A}_\alpha^{-1}$ :

**Theorem 4.1.** *The operator  $\mathcal{A}_\alpha$  defined in (55) defines an isomorphism between  $X \times Y$  and its dual  $X' \times Y'$ . Furthermore*

$$(56) \quad \|\mathcal{A}_\alpha\|_{\mathcal{L}(X \times Y, X' \times Y')} \leq c_5 \quad \text{and} \quad \|\mathcal{A}_\alpha^{-1}\|_{\mathcal{L}(X' \times Y', X \times Y)} \leq c_6 \frac{1}{\alpha},$$

where  $c_5, c_6 > 0$  do not depend on the regularization parameter  $\alpha$  or on the mesh size  $h$ .

This theorem indicates that the boundedness of  $\mathcal{A}_\alpha^{-1}$  deteriorates as  $\alpha \rightarrow 0$ , which is consistent with the ill-posed nature of the underlying PDE constrained optimization problem.

**4.3. An auxiliary operator.** Our analysis of the eigenvalue distribution of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$  is based on the properties of the auxiliary operator

$$(57) \quad \widehat{\mathcal{A}}_\alpha = \begin{bmatrix} \alpha L & 0 & B' \\ 0 & K & A' + \frac{1}{\alpha} K' \\ B & A + \frac{1}{\alpha} K & 0 \end{bmatrix},$$

which we may write in the form

$$\widehat{\mathcal{A}}_\alpha = \begin{bmatrix} M_\alpha & \widehat{N}' \\ \widehat{N} & 0 \end{bmatrix},$$

where

$$(58) \quad \widehat{N} = \begin{bmatrix} B & A + \frac{1}{\alpha} K \end{bmatrix} : X \rightarrow Y'$$

and  $M_\alpha$  is defined in (48).

For  $\widehat{\mathcal{A}}_\alpha$  we can prove an inf-sup condition which is independent of  $\alpha$ .

**Lemma 4.4.** *Assume that assumptions A1-A6 hold. Then there exists a constant  $c_{12} > 0$ , independent of  $\alpha$  and  $h$ , such that*

$$\inf_{y \in Y} \sup_{x \in X} \frac{\langle \widehat{N}x, y \rangle}{\|x\|_X \|y\|_Y} \geq c_{12}.$$

*Proof.* As in the proof of Lemma 4.3, let  $y \in Y$  be arbitrary and note that

$$\hat{x} = (0, y) \in X.$$

Clearly

$$\sup_{x \in X} \frac{\langle \widehat{N}x, y \rangle}{(\|x\|_X \|y\|_Y)} \geq \frac{\langle \widehat{N}\hat{x}, y \rangle}{(\|\hat{x}\|_X \|y\|_Y)}$$

and

$$\begin{aligned} \langle \widehat{N}\hat{x}, y \rangle &= \langle B0, y \rangle + \langle Ay, y \rangle + \langle \frac{1}{\alpha} Ky, y \rangle \\ &= \langle Ay, y \rangle + \frac{1}{\alpha} \langle Ky, y \rangle \\ &\geq c_1 \|y\|_{H_2}^2 + \frac{1}{\alpha} \|Ty\|_{H_3}^2, \end{aligned}$$

cf. inequality (5) and the definition (14) of  $K$ .

Note that

$$\begin{aligned} \|\hat{x}\|_X^2 \|y\|_Y^2 &= \|(0, y)\|_X^2 \|y\|_Y^2 \\ &= [\alpha \|y\|_{H_2}^2 + \|Ty\|_{H_3}^2] \left[ \frac{1}{\alpha} \|y\|_{H_2}^2 \right] \\ &= \|y\|_{H_2}^4 + \frac{1}{\alpha} \|Ty\|_{H_3}^2 \|y\|_{H_2}^2 \\ &\leq \|y\|_{H_2}^4 + \frac{1}{\alpha} \|Ty\|_{H_3}^2 \|y\|_{H_2}^2 + \frac{1}{\alpha^2} \|Ty\|_{H_3}^4 \\ &\leq \left( \|y\|_{H_2}^2 + \frac{1}{\alpha} \|Ty\|_{H_3}^2 \right)^2. \end{aligned}$$

We can thus conclude that

$$\begin{aligned} \frac{\langle N\hat{x}, y \rangle}{\|\hat{x}\|_X \|y\|_Y} &\geq \frac{c_1 \|y\|_{H_2}^2 + \frac{1}{\alpha} \|Ty\|_{H_3}^2}{\|y\|_{H_2}^2 + \frac{1}{\alpha} \|Ty\|_{H_3}^2} \\ &\geq \min\{c_1, 1\}. \end{aligned}$$

Finally, because  $y \in Y$  was arbitrary, it follows that

$$\inf_{y \in Y} \sup_{x \in X} \frac{\langle \hat{N}x, y \rangle}{\|x\|_X \|y\|_Y} \geq c_{12}.$$

□

On the other hand,  $\hat{N}$  is not bounded independently of the regularization parameter  $\alpha$ :

$$\begin{aligned} |\langle \hat{N}x, y \rangle| &\leq |\langle Bx_1, y \rangle| + |\langle Ax_2, y \rangle| + \frac{1}{\alpha} |(Tx_2, Ty)_{H_3}| \\ &\leq \|B\| \|x_1\|_{H_1} \|y\|_{H_2} + \|A\| \|x_2\|_{H_2} \|y\|_{H_2} + \frac{1}{\alpha} \|T\|^2 \|x_2\|_{H_2} \|y\|_{H_2} \\ &= \|B\| \sqrt{\alpha} \|x_1\|_{H_1} \frac{1}{\sqrt{\alpha}} \|y\|_{H_2} + \left( \|A\| + \frac{1}{\alpha} \|T\|^2 \right) \sqrt{\alpha} \|x_2\|_{H_2} \frac{1}{\sqrt{\alpha}} \|y\|_{H_2} \\ &\leq \|B\| \|x\|_X \|y\|_Y + \left( \|A\| + \frac{1}{\alpha} \|T\|^2 \right) \|x\|_X \|y\|_Y \\ &= \left( \|B\| + \|A\| + \frac{1}{\alpha} \|T\|^2 \right) \|x\|_X \|y\|_Y. \end{aligned}$$

From these considerations and Theorem 1.1 on page 42 in [21] we obtain the following lemma:

**Lemma 4.5.** *There exist constants  $C_2$  and  $C_3$ , which are independent of  $h$  and  $\alpha$ , such that*

$$\|\widehat{\mathcal{A}}_\alpha\|_{\mathcal{L}(X \times Y, X' \times Y')} \leq C_2 \frac{1}{\alpha} \quad \text{and} \quad \|\widehat{\mathcal{A}}_\alpha^{-1}\|_{\mathcal{L}(X' \times Y', X \times Y)} \leq C_3,$$

where  $\widehat{\mathcal{A}}_\alpha$  is the operator defined in (57).

**4.4. Distribution of eigenvalues.** We will now analyze the distribution of the eigenvalues of the preconditioned operator  $\mathcal{B}_\alpha \mathcal{A}_\alpha$  and use it to characterize the convergence properties of the minimal residual method.

Recall the definition (24) of the preconditioner  $\mathcal{B}_\alpha$  and the definition (18)-(21) of the Hilbert spaces  $X$  and  $Y$ . It follows that  $\mathcal{B}_\alpha$  defines an isomorphism between  $X' \times Y'$  and  $X \times Y$  and that

$$(59) \quad \|\mathcal{B}_\alpha\|_{\mathcal{L}(X' \times Y', X \times Y)} \leq c_7 \quad \text{and} \quad \|\mathcal{B}_\alpha^{-1}\|_{\mathcal{L}(X \times Y, X' \times Y')} \leq c_8,$$

where  $c_7, c_8 > 0$  are independent of  $\alpha$  and  $h$ .

We now use (56) and (59) to bound the eigenvalues of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$ :

**Theorem 4.2.** *Let  $\mathcal{A}_\alpha$  and  $\mathcal{B}_\alpha$  be the operators defined in (55) and (24), respectively. The eigenvalues of the preconditioned operator  $\mathcal{B}_\alpha \mathcal{A}_\alpha$  satisfy the bound*

$$c_9 \alpha \leq |\lambda| \leq c_{10} \quad \text{for all } \lambda \in \text{sp}(\mathcal{B}_\alpha \mathcal{A}_\alpha),$$

where  $c_9, c_{10} > 0$  do not depend on  $\alpha$  or  $h$  and  $\text{sp}(\mathcal{B}_\alpha \mathcal{A}_\alpha)$  denotes the spectrum of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$ .

*Proof.* Assume that  $\lambda, q$  is an eigenvalue, eigenvector pair of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$ , i.e.

$$\mathcal{B}_\alpha \mathcal{A}_\alpha q = \lambda q.$$

Then

$$\|\mathcal{B}_\alpha \mathcal{A}_\alpha q\|_{X \times Y} = |\lambda| \|q\|_{X \times Y}$$

and

$$|\lambda| = \frac{\|\mathcal{B}_\alpha \mathcal{A}_\alpha q\|_{X \times Y}}{\|q\|_{X \times Y}} \leq \frac{\|\mathcal{B}_\alpha\|_{\mathcal{L}(X' \times Y', X \times Y)} \|\mathcal{A}_\alpha\|_{\mathcal{L}(X \times Y, X' \times Y')} \|q\|_{X \times Y}}{\|q\|_{X \times Y}} \leq c_7 c_5,$$

where the last inequality follows from (59) and (56).

Furthermore, since

$$\frac{1}{\lambda} q = \mathcal{A}_\alpha^{-1} \mathcal{B}_\alpha^{-1} q,$$

we find that

$$\begin{aligned} \frac{1}{|\lambda|} \|q\|_{X \times Y} &= \|\mathcal{A}_\alpha^{-1} \mathcal{B}_\alpha^{-1} q\|_{X \times Y} \\ &\leq \|\mathcal{A}_\alpha^{-1}\|_{\mathcal{L}(X' \times Y', X \times Y)} \|\mathcal{B}_\alpha^{-1}\|_{\mathcal{L}(X \times Y, X' \times Y')} \|q\|_{X \times Y} \\ &\leq c_6 \frac{1}{\alpha} c_8 \|q\|_{X \times Y}. \end{aligned}$$

Consequently

$$|\lambda| \geq \frac{\alpha}{c_6 c_8},$$

which finishes the proof.  $\square$

From Theorem 4.2 we find that the spectral condition number  $\kappa(\mathcal{B}_\alpha \mathcal{A}_\alpha)$  of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$  is of order  $O(\alpha^{-1})$ :

**Corollary 4.1.** *Let  $\mathcal{A}_\alpha$  and  $\mathcal{B}_\alpha$  be the operators defined in (55) and (24), respectively. Then*

$$\kappa(\mathcal{B}_\alpha \mathcal{A}_\alpha) = \frac{\max_{\lambda \in \text{sp}(\mathcal{B}_\alpha \mathcal{A}_\alpha)} |\lambda|}{\min_{\lambda \in \text{sp}(\mathcal{B}_\alpha \mathcal{A}_\alpha)} |\lambda|} \leq \frac{c_{11}}{\alpha}.$$

This bound is certainly consistent with the results reported in Table 5. (The condition numbers associated with the problem studied in Example 1 increase slower as  $\alpha$  decreases, see Table 2). In view of the numbers presented in tables 1 and 4, Corollary 4.1 and the classical error estimate for the minimal residual method (32) provide a pessimistic bound for the number of Krylov subspace iterations needed to solve (23). A more careful study of the distribution of the eigenvalues of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$  is required.

The next step in our analysis is to write  $\mathcal{B}_\alpha \mathcal{A}_\alpha$  in the form

$$(60) \quad \mathcal{B}_\alpha \mathcal{A}_\alpha = \mathcal{B}_\alpha \left( \widehat{\mathcal{A}}_\alpha + \mathcal{K}_\alpha \right) = \mathcal{B}_\alpha \widehat{\mathcal{A}}_\alpha + \mathcal{B}_\alpha \mathcal{K}_\alpha,$$

where  $\widehat{\mathcal{A}}_\alpha$  is defined in (57) and

$$(61) \quad \mathcal{K}_\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\alpha} K' \\ 0 & -\frac{1}{\alpha} K & 0 \end{bmatrix}.$$

More precisely, first we show that the absolute value of the eigenvalues of  $\mathcal{B}_\alpha \widehat{\mathcal{A}}_\alpha$  is bounded below independently of  $\alpha$  and  $h$ . Thereafter the decay of



the eigenvalues of  $\mathcal{B}_\alpha \mathcal{K}_\alpha$  is discussed, and finally we use a theorem due to Hermann Weyl to reveal the basic properties of the spectrum of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$ .

4.4.1. *Eigenvalues of  $\mathcal{B}_\alpha \widehat{\mathcal{A}}_\alpha$ .* Recall that

$$\|\mathcal{B}_\alpha\|_{\mathcal{L}(X' \times Y', X \times Y)}, \|\mathcal{B}_\alpha^{-1}\|_{\mathcal{L}(X \times Y, X' \times Y')} \text{ and } \|\widehat{\mathcal{A}}_\alpha^{-1}\|_{\mathcal{L}(X' \times Y', X \times Y)}$$

are bounded independently of  $h$  and  $\alpha$  and that

$$\|\widehat{\mathcal{A}}_\alpha\|_{\mathcal{L}(X' \times Y', X \times Y)} \leq C_2 \alpha^{-1},$$

cf. Lemma 4.5. Therefore, analogous to the arguments shown in the proof of Theorem 4.2, we can obtain the following lemma:

**Lemma 4.6.** *The eigenvalues of the operator  $\mathcal{B}_\alpha \widehat{\mathcal{A}}_\alpha$  satisfy the bound*

$$C_4 \leq |\gamma| \leq C_5 \frac{1}{\alpha} \quad \text{for all } \gamma \in \text{sp}(\mathcal{B}_\alpha \widehat{\mathcal{A}}_\alpha),$$

where  $\mathcal{B}_\alpha$  and  $\widehat{\mathcal{A}}_\alpha$  are the operators defined in (24) and (57), respectively. Here,  $C_4$  and  $C_5$  are positive constants that do not depend on  $\alpha$  or  $h$ .

Note the difference between Theorem 4.2 and Lemma 4.6: The absolute value of the eigenvalues of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$  are bounded above independently of  $\alpha$ , whereas the absolute value of the eigenvalues of  $\mathcal{B}_\alpha \widehat{\mathcal{A}}_\alpha$  are bounded below independently of  $\alpha$ . These facts turn out to be crucial in our analysis.

4.4.2. *Eigenvalues of  $\mathcal{B}_\alpha \mathcal{K}_\alpha$ .* We now turn our attention to the properties of  $\mathcal{B}_\alpha \mathcal{K}_\alpha$ . From the definition (14) of  $K$  we find that  $K' = K$ , where

$$K' : H_2 \rightarrow H'_2, \quad u \rightarrow (T\phi, Tu)_{H_3} = (T^*Tu, \phi)_{H_2} \quad \forall \phi \in H_2.$$

This means that

$$\begin{aligned} \mathcal{B}_\alpha \mathcal{K}_\alpha &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -(\alpha Q_2 + K)^{-1} \frac{1}{\alpha} K \\ 0 & -\alpha Q_2^{-1} \frac{1}{\alpha} K & 0 \end{bmatrix} \\ (62) \quad &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -(\alpha Q_2 + K)^{-1} \frac{1}{\alpha} K \\ 0 & -Q_2^{-1} K & 0 \end{bmatrix}, \end{aligned}$$

see (24) and (61). We will now characterize the properties of  $Q_2^{-1}K : H_2 \rightarrow H_2$  and  $(\alpha Q_2 + K)^{-1}K : H_2 \rightarrow H_2$  and use that information to explore  $\mathcal{B}_\alpha \mathcal{K}_\alpha$ .

Please note that all the eigenvalues of  $Q_2^{-1}K$  must be larger or equal to zero: If  $\tilde{\sigma}_i, x_i$  is an eigenvalue, eigenfunction pair of  $Q_2^{-1}K$ , i.e.

$$(63) \quad Q_2^{-1}K x_i = \tilde{\sigma}_i x_i,$$

then

$$K x_i = \tilde{\sigma}_i Q_2 x_i$$

which implies that

$$\langle K x_i, x_i \rangle = \tilde{\sigma}_i \langle Q_2 x_i, x_i \rangle.$$

From (25) and the fact that  $\langle K x, x \rangle \geq 0$  for all  $x \in H_2$  we find that  $\tilde{\sigma}_i$  must be non-negative.

Next,

$$(\alpha I + Q_2^{-1}K)x_i = (\alpha + \tilde{\sigma}_i)x_i,$$

and hence

$$(64) \quad (\alpha I + Q_2^{-1}K)^{-1}x_i = \frac{1}{\alpha + \tilde{\sigma}_i}x_i.$$

Clearly

$$(\alpha Q_2 + K)^{-1} = [Q_2(\alpha I + Q_2^{-1}K)]^{-1} = (\alpha I + Q_2^{-1}K)^{-1}Q_2^{-1},$$

which together with (63) and (64) imply that

$$(\alpha Q_2 + K)^{-1}Kx_i = [(\alpha I + Q_2^{-1}K)^{-1}][Q_2^{-1}K]x_i = \frac{\tilde{\sigma}_i}{\alpha + \tilde{\sigma}_i}x_i \quad \text{for } i = 1, 2, \dots, \nu_2,$$

and furthermore

$$(65) \quad [(\alpha Q_2 + K)^{-1}K][Q_2^{-1}K]x_i = \frac{\tilde{\sigma}_i^2}{\alpha + \tilde{\sigma}_i}x_i \quad \text{for } i = 1, 2, \dots, \nu_2.$$

Here,  $\tilde{\sigma}_i, x_i$  for  $i = 1, 2, \dots, \nu_2$  are the eigenvalue, eigenfunction pairs of  $Q_2^{-1}K$ .

We are now in the position to find formulas for the eigenvalues of  $\mathcal{B}_\alpha \mathcal{K}_\alpha$ . Assume that

$$\mathcal{B}_\alpha \mathcal{K}_\alpha \begin{bmatrix} v_i \\ u_i \\ w_i \end{bmatrix} = \beta_i \begin{bmatrix} v_i \\ u_i \\ w_i \end{bmatrix},$$

that is, see (62),

$$\begin{aligned} -\frac{1}{\alpha}(\alpha Q_2 + K)^{-1}Kw_i &= \beta_i u_i \\ -Q_2^{-1}Ku_i &= \beta_i w_i, \end{aligned}$$

or

$$\begin{aligned} [(\alpha Q_2 + K)^{-1}K][Q_2^{-1}K]u_i &= \alpha \beta_i^2 u_i \\ w_i &= -\frac{1}{\beta_i}Q_2^{-1}Ku_i, \end{aligned}$$

provided that  $\beta_i \neq 0$ . By invoking (65) and (63) we find that

$$u_i = x_i, \quad w_i = -\frac{\tilde{\sigma}_i}{\beta_i}x_i, \quad \text{and} \quad \alpha \beta_i^2 = \frac{\tilde{\sigma}_i^2}{\alpha + \tilde{\sigma}_i}.$$

Consequently,

$$\beta_i^2 = \frac{1}{\alpha} \left( \frac{\tilde{\sigma}_i^2}{\alpha + \tilde{\sigma}_i} \right) \leq \frac{\tilde{\sigma}_i^2}{\alpha^2},$$

where we have used the fact that  $\tilde{\sigma}_i \geq 0$ . We hence conclude that

$$(66) \quad |\beta_i| \leq \frac{\tilde{\sigma}_i}{\alpha} \quad \text{for } i = 1, 2, \dots, \nu_2.$$

That is,  $\nu_2$  of the eigenvalues of  $\mathcal{B}_\alpha \mathcal{K}_\alpha$  are “inherited” from  $Q_2^{-1}K$ , and the dimension of the null space of  $\mathcal{B}_\alpha \mathcal{K}_\alpha$  is at least  $\nu_1 + \nu_2$ , see (45)-(47).

Since  $\tilde{\sigma}_i$  is an eigenvalue of  $Q_2^{-1}K$  we need to understand more about this operator in order to fully characterize the spectrum of  $\mathcal{B}_\alpha \mathcal{K}_\alpha$ . To this end, recall the definition (14) of  $K$  and that, in the infinite dimensional case, 0 is a cluster point for the eigenvalues of  $T^*T : H_2 \rightarrow H_2$ , cf. Section 2. Clearly,

$$K = J_2 T^* T,$$

where  $J_2 : H_2 \rightarrow H_2'$  denotes the Riesz map. Note that

$$Q_2^{-1}K = (Q_2^{-1}J_2)(J_2^{-1}K) = (Q_2^{-1}J_2)(T^*T)$$

and that  $(Q_2^{-1}J_2) : H_2 \rightarrow H_2$  is bounded and continuously invertible, cf. (24)-(25). In the infinite dimensional case it therefore follows that the ill-posedness of the equation

$$T^*Tx = y$$

is inherited by

$$Q_2^{-1}Kx = Q_2^{-1}J_2y.$$

That is, the decay of the eigenvalues of  $T^*T$  and  $Q_2^{-1}K$  are closely related.

Finite dimensional problems are more subtle since linear operators on finite dimensional spaces always are continuously invertible on the orthogonal complement of their null space. However, it seems reasonable to assume that the discretized operators inherit the basic properties of their continuous counterparts, and we therefore make the following assumption:

**A8:** In the finite dimensional case we assume that the eigenvalues  $\tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \dots \geq \tilde{\sigma}_{\nu_2}$  of  $Q_2^{-1}K : H_2 \rightarrow H_2$  satisfy

$$(67) \quad 0 \leq \tilde{\sigma}_i \leq b_8 e^{-b_9 i} \quad \text{for } i = 1, 2, \dots, \nu_2.$$

Here,  $b_8$  and  $b_9$  are positive constants not depending on  $\nu_2 = \dim(H_2) < \infty$ ,  $h$  or  $\alpha$ .

For mildly ill-posed problems we must of course make an analogous assumption. Further details concerning such cases can be found in Appendix B.

From (66) and Assumption A8 we conclude that the eigenvalues<sup>1</sup>

$$(68) \quad |\tilde{\beta}_1| \geq |\tilde{\beta}_2| \geq \dots \geq |\tilde{\beta}_n|$$

of  $\mathcal{B}_\alpha \mathcal{K}_\alpha$  must satisfy

$$(69) \quad 0 \leq |\tilde{\beta}_i| \leq \frac{1}{\alpha} b_8 e^{-b_9 i} \quad \text{for } i = 1, 2, \dots, n.$$

Here,  $n$  is the dimension of  $X \times Y$ , i.e.  $n = \nu_1 + \nu_2 + \nu_2$ , cf. (45)-(47).

**4.4.3. Eigenvalues of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$ .** We will now employ Theorem 4.2, Lemma 4.6, inequalities (69) and a theorem characterizing the spectrum of sums of self-adjoint<sup>2</sup> operators to reveal the structure of the eigenvalue distribution of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$ . To this end, let

- $\lambda_1 \geq \lambda_2 \geq \dots \lambda_n$  denote the eigenvalues of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$ ,
- $\gamma_1 \geq \gamma_2 \geq \dots \gamma_n$  denote the eigenvalues of  $\mathcal{B}_\alpha \widehat{\mathcal{A}}_\alpha$ ,
- $\beta_1 \geq \beta_2 \geq \dots \beta_n$  denote the eigenvalues of  $\mathcal{B}_\alpha \mathcal{K}_\alpha$ . (In contrast to (68), now sorted with respect to value).

Keep in mind that

$$|\lambda_i| \leq c_{10} \text{ and } |\gamma_i| \geq C_4 \quad \text{for } i = 1, 2, \dots, n,$$

where  $c_{10}$  and  $C_4$  are independent of  $\alpha$  and  $h$ , cf. Theorem 4.2 and Lemma 4.6.

<sup>1</sup>We order these eigenvalues with respect to absolute value.

<sup>2</sup>Please note that  $\mathcal{B}_\alpha \mathcal{A}_\alpha$ ,  $\mathcal{B}_\alpha \widehat{\mathcal{A}}_\alpha$  and  $\mathcal{B}_\alpha \mathcal{K}_\alpha$  are self-adjoint operators with respect to the inner product  $(x, y) = \langle \mathcal{B}_\alpha^{-1}x, y \rangle$ , cf. the discussion of the preconditioner (24).

Inequalities (69) imply that a majority of the eigenvalues of  $\mathcal{B}_\alpha \mathcal{K}_\alpha$  are, in the sense of absolute value, very small. Let  $m$  be the integer such that

$$(70) \quad \dots |\tilde{\beta}_{m-1}| \geq |\tilde{\beta}_m| > \frac{C_4}{2} \geq |\tilde{\beta}_{m+1}| \geq |\tilde{\beta}_{m+2}| \geq \dots,$$

where  $C_4$  is the constant present in Lemma 4.6. In fact, from (69) we obtain the following bound for  $m$  in terms of  $\alpha$

$$(71) \quad m \leq \left\lceil \frac{C_6 - \ln(\alpha)}{b_9} \right\rceil,$$

where  $C_6 = \ln(2) + \ln(b_8) - \ln(C_4)$ . Here,  $\lceil a \rceil$ , for a real number  $a$ , denotes the smallest integer  $\geq a$ .

Recall that  $\mathcal{B}_\alpha \mathcal{A}_\alpha = \mathcal{B}_\alpha \widehat{\mathcal{A}}_\alpha + \mathcal{B}_\alpha \mathcal{K}_\alpha$ . According to [28] (page 255) and/or [51]:

$$(72) \quad \lambda_{i+j-1} \leq \gamma_i + \beta_j \quad \text{for } i + j - 1 \leq n,$$

and, since  $\mathcal{B}_\alpha \widehat{\mathcal{A}}_\alpha = \mathcal{B}_\alpha \mathcal{A}_\alpha - \mathcal{B}_\alpha \mathcal{K}_\alpha$ ,

$$\gamma_{i+j-1} \leq \lambda_j - \beta_{n-i+1} \quad \text{for } i + j - 1 \leq n$$

or

$$(73) \quad \lambda_j \geq \gamma_{i+j-1} + \beta_{n-i+1} \quad \text{for } i + j - 1 \leq n.$$

We are now in the position to prove the following result:

**Theorem 4.3.** *Assume that A1-A6, A8 hold and let*

$$(74) \quad q = 2 \left\lceil \frac{C_6 - \ln(\alpha)}{b_9} \right\rceil.$$

*Then there exist real numbers  $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_q$ , possibly depending on  $\alpha$  and  $h$ , such that the spectrum  $\text{sp}(\mathcal{B}_\alpha \mathcal{A}_\alpha)$  of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$  satisfies*

$$(75) \quad \text{sp}(\mathcal{B}_\alpha \mathcal{A}_\alpha) \subseteq \left( \bigcup_{i=1}^q \{\tilde{\lambda}_i\} \right) \cup [-C_7, -C_8] \cup [C_8, C_7].$$

*The constants  $C_7$  and  $C_8$  are independent of  $\alpha$  and  $h$ .*

*Proof.* First, in Theorem 4.2 we proved that

$$(76) \quad |\lambda_i| \leq c_{10} \quad \text{for } i = 1, 2, \dots, n,$$

and hence we may choose  $C_7 = c_{10}$ .

Let  $l$  be the integer such that

$$(77) \quad \gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_l \geq 0 > \gamma_{l+1} \geq \dots \geq \gamma_n,$$

where  $\{\gamma_i\}_{i=1}^n$  are the eigenvalues of  $\mathcal{B}_\alpha \widehat{\mathcal{A}}_\alpha$ . Note that Lemma 4.6 implies that

$$(78) \quad \gamma_l \geq C_4 \quad \text{and} \quad \gamma_{l+1} \leq -C_4.$$

For  $j = l - i + 1$  it follows from inequality (73) that

$$\lambda_{l-i+1} \geq \gamma_l + \beta_{n-i+1} \quad \text{for } i = 1, 2, \dots, l,$$

and by employing (78):

$$(79) \quad \lambda_{l-i+1} \geq C_4 + \beta_{n-i+1} \quad \text{for } i = 1, 2, \dots, l.$$

Consequently, from (70) we conclude that

$$(80) \quad \text{At most } m \text{ of the eigenvalues } \lambda_1, \lambda_2, \dots, \lambda_l \text{ are smaller than } \frac{C_4}{2}.$$

By choosing  $i = l + 1$  in (72) we find that

$$\lambda_{l+j} \leq \gamma_{l+1} + \beta_j \quad \text{for } l + j \leq n,$$

or

$$\lambda_{l+j} \leq -C_4 + \beta_j \quad \text{for } j = 1, \dots, n - l,$$

where we have used (78). Hence, (70) yields that

$$(81) \quad \text{At most } m \text{ of the eigenvalues } \lambda_{l+1}, \lambda_2, \dots, \lambda_n \text{ are larger than } -\frac{C_4}{2}.$$

From (76), (80) and (81) it follows that at most  $2m$  of the eigenvalues of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$  are outside the set  $[-C_7, -C_4/2] \cup [C_4/2, C_7]$ . The theorem is now a consequence of the bound (71) for  $m$ .  $\square$

Thus, the number of eigenvalues outside the set  $[-C_7, -C_8] \cup [C_8, C_7]$  grows at most logarithmically as  $\alpha$  decreases. We will refer to  $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_q$  as isolated eigenvalues. This is certainly in agreement with what we observed in figures 2 and 6.

**4.5. Convergence properties.** Convergence properties of iterative schemes for linear systems with isolated eigenvalues have been analyzed by, among others, Andersson, Axelsson and Lindskog [2, 6, 7]. However, as far as the authors know, such results have so far not been presented for the minimal residual method. To prove the main result of this paper we combine techniques from Axelsson and Lindskog [6, 7] and Hackbusch [31]. The following theorem states that the order of the number of iterations needed cannot be any worse than  $O((\ln(\alpha))^2)$ , provided that the convergence is measured in the energy norm.

**Theorem 4.4.** *Let  $p^*$  denote the solution of (23), assume that assumptions A1-A6 and A8 hold, and let  $\epsilon > 0$  be a given error tolerance. If*

$$k \geq \frac{2}{\ln(\eta^{-1})} \left\{ \ln \left( \frac{2}{\epsilon} \right) + 2 \left\lceil \frac{C_6 - \ln(\alpha)}{b_9} \right\rceil (\ln(c_{17}) - \ln(\alpha)) \right\} \\ + 2 \left\lceil \frac{C_6 - \ln(\alpha)}{b_9} \right\rceil + 4$$

then

$$\frac{\|p_k - p^*\|_{E,\alpha}}{\|p_0 - p^*\|_{E,\alpha}} = \left( \frac{\langle \mathcal{A}_\alpha(p_k - p^*), \mathcal{B}_\alpha \mathcal{A}_\alpha(p_k - p^*) \rangle}{\langle \mathcal{A}_\alpha(p_0 - p^*), \mathcal{B}_\alpha \mathcal{A}_\alpha(p_0 - p^*) \rangle} \right)^{1/2} \leq \epsilon$$

where  $p_k$  is the  $k$ th approximation of  $p^*$  generated by the minimal residual method applied to (23). The constants  $C_6, b_9, c_{17}$  and  $\eta$  do not depend on the regularization parameter  $\alpha$  or on the mesh size  $h$ .

*Proof.* According to Hackbusch [31], see page 287,

$$(82) \quad \frac{\|p_k - p^*\|_{E,\alpha}}{\|p_0 - p^*\|_{E,\alpha}} \leq \min_{\Phi_k \in \bar{\Pi}_k} \max_{\lambda \in \text{SP}(\mathcal{B}_\alpha \mathcal{A}_\alpha)} |\Phi_k(\lambda)|,$$

where  $\bar{\Pi}_k$  is the set of all polynomials of degree  $\leq k$  with  $\Phi_k(0) = 1$ .

Recall the distribution (75) of the eigenvalues of  $\mathcal{B}_\alpha \mathcal{A}_\alpha$ . Let

$$\begin{aligned} l &= \lceil k/2 \rceil - 1, \\ c_{14} &= C_8^2, \\ c_{15} &= C_7^2, \end{aligned}$$

and consider the polynomial

$$\Phi_{l-\lceil q/2 \rceil}^*(x; c_{14}, c_{15}) = \frac{T_{l-\lceil q/2 \rceil} \left( \frac{c_{15} + c_{14} - 2x}{c_{15} - c_{14}} \right)}{T_{l-\lceil q/2 \rceil} \left( \frac{c_{15} + c_{14}}{c_{15} - c_{14}} \right)},$$

where  $q$  is the number of isolated eigenvalues in (75) and  $T_{l-\lceil q/2 \rceil}$  is the Chebyshev polynomial of order  $l - \lceil q/2 \rceil$ . It is well known that, see e.g. Axelsson and Lindskog [7] and references therein,

$$(83) \quad \max_{x \in [c_{14}, c_{15}]} |\Phi_{l-\lceil q/2 \rceil}^*(x; c_{14}, c_{15})| = 2 \frac{\eta^{l-\lceil q/2 \rceil}}{1 + \eta^{2l-2\lceil q/2 \rceil}} \leq 2\eta^{l-\lceil q/2 \rceil},$$

where

$$\eta = \left( 1 - \sqrt{\frac{c_{14}}{c_{15}}} \right) / \left( 1 + \sqrt{\frac{c_{14}}{c_{15}}} \right) \in (0, 1).$$

Please observe that

$$\lambda \in [-C_7, -C_8] \cup [C_8, C_7] \Rightarrow \lambda^2 \in [C_8^2, C_7^2] = [c_{14}, c_{15}]$$

and that  $\Phi_{l-\lceil q/2 \rceil}^*(\lambda^2; c_{14}, c_{15})$  is of degree  $2l - 2\lceil q/2 \rceil \leq k - q$ . Consequently, the polynomial

$$\Psi_k(\lambda) = \left[ \prod_{i=1}^q \left( 1 - \frac{\lambda}{\tilde{\lambda}_i} \right) \right] \Phi_{l-\lceil q/2 \rceil}^*(\lambda^2; c_{14}, c_{15})$$

is in  $\overline{\Pi}_k$  and satisfies

$$\Psi_k(\tilde{\lambda}_i) = 0 \quad \text{for } i = 1, 2, \dots, q.$$

Furthermore, from (83) we find that

$$\begin{aligned} \max_{\lambda \in \text{sp}(\mathcal{B}_\alpha \mathcal{A}_\alpha)} |\Psi_k(\lambda)| &\leq \max_{\lambda \in (\cup_{i=1}^q \tilde{\lambda}_i) \cup [-C_7, -C_8] \cup [C_8, C_7]} |\Psi_k(\lambda)| \\ &= \max_{\lambda \in [-C_7, -C_8] \cup [C_8, C_7]} |\Psi_k(\lambda)| \\ &\leq \left[ \prod_{i=1}^q \max_{\lambda \in [-C_7, C_7]} \left| 1 - \frac{\lambda}{\tilde{\lambda}_i} \right| \right] \max_{\lambda^2 \in [c_{14}, c_{15}]} |\Phi_{l-\lceil q/2 \rceil}^*(\lambda^2; c_{14}, c_{15})| \\ &\leq 2 \left[ \prod_{i=1}^q \frac{c_{16}}{|\tilde{\lambda}_i|} \right] \eta^{l-\lceil q/2 \rceil} \\ &\leq 2 \left( \frac{c_{16}}{c_9 \alpha} \right)^q \eta^{\lceil k/2 \rceil - 1 - \lceil q/2 \rceil}, \end{aligned}$$

where the last inequality follows from Theorem 4.2 and  $c_{16} = 2C_7$ .

Clearly,

$$2 \left( \frac{c_{16}}{c_9 \alpha} \right)^q \eta^{\lceil k/2 \rceil - 1 - \lceil q/2 \rceil} \leq \epsilon$$

if

$$\begin{aligned}
k &\geq \frac{2}{\ln(\eta^{-1})} \left( \ln \left( \frac{2}{\epsilon} \right) + q(\ln(c_{16}) - \ln(c_9\alpha)) \right) + q + 4 \\
&= \frac{2}{\ln(\eta^{-1})} \left\{ \ln \left( \frac{2}{\epsilon} \right) + 2 \left\lceil \frac{C_6 - \ln(\alpha)}{b_9} \right\rceil (\ln(c_{16}) - \ln(c_9\alpha)) \right\} \\
(84) \quad &+ 2 \left\lceil \frac{C_6 - \ln(\alpha)}{b_9} \right\rceil + 4,
\end{aligned}$$

where (74) is used in the last equality. Since  $\Psi_k \in \bar{\Pi}_k$  the theorem is now a consequence of (82).  $\square$

Theorem 4.4 was derived assuming that the eigenvalues of  $T^*T$  satisfy (7), i.e. we considered the severely ill-posed case. Our argument can be modified in a rather straightforward manner to also cover mildly ill-posed problems. That is, to cases in which  $\mathcal{A}6$  is replaced by  $\mathcal{A}7$ . In such situations the number of iterations needed by the minimal residual method will not grow faster than  $O(-\alpha^{-1/\tilde{\xi}} \ln(\alpha))$ . The result is thus somewhat weaker for mildly ill-posed problems, which might surprise some readers. Further details about this topic can be found in Appendix B.

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## APPENDIX A

In this section we use the notation introduced in Section 3.2 for Example 2. Our goal is to prove that  $\mathcal{R}(\tilde{A}^{-1}\tilde{B}) = W$  and that  $\tilde{A}^{-1}\tilde{B}$  is a one-to-one mapping. To this end, let us analyze the bilinear form

$$a(\psi, \phi) = \int_H \left( \mathbf{M}_i \nabla(\tilde{R}\psi) \right) \cdot \nabla(\tilde{R}\phi) dx \quad \text{for } \psi, \phi \in \tilde{H}^1(P)$$

associated with the right hand side of the state equation (41). Since  $H \subset P$ , assumptions (36)-(38) imply that  $a(\cdot, \cdot)$  is continuous.

In the first part of this proof we will show that  $a(\cdot, \cdot)$  is coercive on  $W \times W \subset \tilde{H}^1(P) \times \tilde{H}^1(P)$ . Let  $w \in W$  be arbitrary. From the definition (44) of  $W$  it follows that  $q = w|_G$  is the weak solution of

$$\begin{aligned}
\nabla \cdot (\mathbf{M}\nabla q) &= 0 \quad \text{in } G, \\
q &= E\tilde{R}w \quad \text{on } \partial H, \\
(\mathbf{M}\nabla q) \cdot \mathbf{n} &= 0 \quad \text{on } \partial P,
\end{aligned}$$

where  $\mathbf{n}$  is the outwards directed normal vector of unit length of  $\partial P$  and  $E : H^1(H) \rightarrow H^{1/2}(\partial H)$  is the trace operator. It thus follows that  $\bar{q} = \bar{w}|_G$ , where

$$\bar{w} = w - \frac{1}{|H|} \int_H w dx,$$

is the weak solution of

$$\begin{aligned}\nabla \cdot (\mathbf{M}\nabla\bar{q}) &= 0 \quad \text{in } G, \\ \bar{q} &= E\tilde{R}\bar{w} \quad \text{on } \partial H, \\ (\mathbf{M}\nabla\bar{q}) \cdot \mathbf{n} &= 0 \quad \text{on } \partial P.\end{aligned}$$

Classical stability estimates for elliptic PDEs, the trace theorem and Poincaré's inequality imply that

$$\|\bar{q}\|_{H^1(G)} \leq \|E\tilde{R}\bar{w}\|_{H^{1/2}(\partial H)} \leq \|E\| \|\bar{w}\|_{H^1(H)} \leq C\|E\| |\bar{w}|_{H^1(H)},$$

from which we conclude that

$$|w|_{H^1(G)} \leq C\|E\| |w|_{H^1(H)}.$$

The coercivity of  $a(\cdot, \cdot)$  on  $W \times W$  is now a consequence of (36)-(38).

In the second part of this proof we consider an arbitrary  $\tau \in W$ . Assumptions (36)-(38) assert that

$$\phi \rightarrow - \int_P (\mathbf{M}\nabla\tau) \cdot \nabla\phi \, dx$$

is a bounded linear functional on  $W$ . Standard theory therefore implies that the following problem has a unique solution: Find  $s \in W$  such that

$$(85) \quad \int_H (\mathbf{M}_i\nabla(\tilde{R}s)) \cdot \nabla(\tilde{R}\phi) \, dx = - \int_P (\mathbf{M}\nabla\tau) \cdot \nabla\phi \, dx \quad \text{for all } \phi \in W.$$

Furthermore, since  $\tilde{H}^1(P) = W \oplus V$ , see (42), (43) and (44), every  $\phi \in \tilde{H}^1(P)$  can be written in a unique way as  $\phi = \phi_W + \phi_V$ , where  $\phi_W \in W$  and  $\phi_V \in V$ . Since  $\phi_V = 0$  on the heart  $H$ , it follows that

$$\int_H (\mathbf{M}_i\nabla(\tilde{R}s)) \cdot \nabla(\tilde{R}\phi) \, dx = \int_H (\mathbf{M}_i\nabla(\tilde{R}s)) \cdot \nabla(\tilde{R}\phi_W) \, dx \quad \text{for all } \phi \in \tilde{H}^1(P).$$

Recall that  $\tau \in W = V^\perp$ , and therefore

$$\int_P (\mathbf{M}\nabla\tau) \cdot \nabla\phi \, dx = \int_P (\mathbf{M}\nabla\tau) \cdot \nabla\phi_W \, dx \quad \text{for all } \phi \in \tilde{H}^1(P).$$

Equation (85) is thus not only valid for all  $\phi \in W$ , but for all  $\phi \in \tilde{H}^1(P)$ ;

$$(86) \quad \int_H (\mathbf{M}_i\nabla(\tilde{R}s)) \cdot \nabla(\tilde{R}\phi) \, dx = - \int_P (\mathbf{M}\nabla\tau) \cdot \nabla\phi \, dx \quad \text{for all } \phi \in \tilde{H}^1(P).$$

This means that  $\tilde{A}^{-1}\tilde{B}\tilde{R}s = \tau$ , and, due to the fact that  $\tau \in W$  was arbitrary, we conclude that  $\mathcal{R}(\tilde{A}^{-1}\tilde{B}) = W$ . The one-to-one property of  $\tilde{A}^{-1}\tilde{B} : H^1(H) \rightarrow W$  follows from the uniqueness of the solution of (85).

## APPENDIX B

This appendix is devoted to mildly ill-posed cases. If  $\mathcal{A}6$  is replaced by  $\mathcal{A}7$  then the integer  $m$  in (70) will become larger. Consequently, the number  $q$  of isolated eigenvalues of  $\mathcal{B}_\alpha\mathcal{A}_\alpha$  in Theorem 4.3 will grow. Our result for mildly ill-posed problems is therefore weaker than for severely ill-posed cases.



The details are as follows. If  $\mathcal{A}7$  holds then we may use (8) and an argument similar to that presented in Section 4.4.2 to conclude that the eigenvalues

$$|\tilde{\beta}_1| \geq |\tilde{\beta}_2| \geq \dots \geq |\tilde{\beta}_n|$$

of  $\mathcal{B}_\alpha \mathcal{K}_\alpha$  must satisfy

$$0 \leq |\tilde{\beta}_i| \leq \frac{1}{\alpha} b_{10} i^{-\tilde{\xi}} \quad \text{for } i = 1, 2, \dots, n.$$

This gives us the following bound for the integer  $m$  defined in (70):

$$m \leq \lceil b_{11} \alpha^{-1/\tilde{\xi}} \rceil,$$

where

$$b_{11} = \left( \frac{2b_{10}}{C_4} \right)^{1/\tilde{\xi}}.$$

The formula for the integer  $q$  in Theorem 4.3 therefore reads

$$(87) \quad q = 2 \lceil b_{11} \alpha^{-1/\tilde{\xi}} \rceil.$$

Please note that (75) still holds, but with  $q$  given by (87).

Concerning the proof of Theorem 4.4, the only modification needed occurs in connection with (84). More specifically, from (87) it follows that: If

$$(88) \quad \begin{aligned} k &\geq \frac{2}{\ln(\eta^{-1})} \left( \ln \left( \frac{2}{\epsilon} \right) + q(\ln(c_{16}) - \ln(c_9 \alpha)) \right) + q + 4 \\ &= \frac{2}{\ln(\eta^{-1})} \left\{ \ln \left( \frac{2}{\epsilon} \right) + 2 \lceil b_{11} \alpha^{-1/\tilde{\xi}} \rceil (\ln(c_{16}) - \ln(c_9 \alpha)) \right\} \\ &\quad + 2 \lceil b_{11} \alpha^{-1/\tilde{\xi}} \rceil + 4, \end{aligned}$$

then

$$\frac{\|p_k - p^*\|_{E,\alpha}}{\|p_0 - p^*\|_{E,\alpha}} = \left( \frac{\langle \mathcal{A}_\alpha(p_k - p^*), \mathcal{B}_\alpha \mathcal{A}_\alpha(p_k - p^*) \rangle}{\langle \mathcal{A}_\alpha(p_0 - p^*), \mathcal{B}_\alpha \mathcal{A}_\alpha(p_0 - p^*) \rangle} \right)^{1/2} \leq \epsilon,$$

The number of iterations needed for mildly ill-posed problems, by the minimal residual method, can therefore not grow faster than of order  $O(-\alpha^{-1/\tilde{\xi}} \ln(\alpha))$ .

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