Reduced basis modeling of complex flow systems

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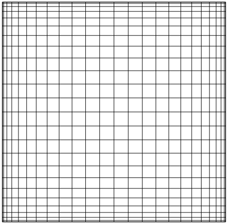
Center for Biomedical Computing

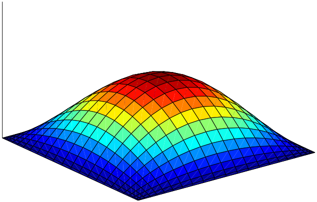
Reduced basis modeling of complex flow systems

Outline

- The reduced basis method
- The reduced basis element method
- Offline/online decoupling: Empirical interpolation
- Multi-block flow systems
- Summary

The reduced basis method



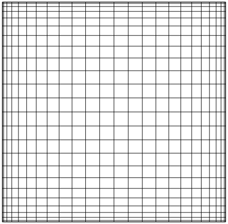


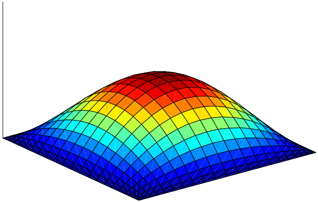
Main idea

Assume a parameter dependent problem

$$F(u; \mu) = 0.$$

The reduced basis method





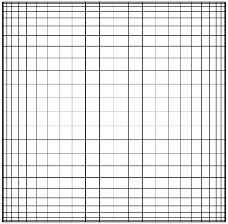
Main idea

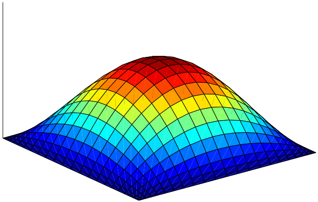
Assume a parameter dependent problem

$$F(u; \mu) = 0.$$

For small changes in the parameter μ, the corresponding solution u often varies in a smooth fashion.
We construct a set of basis functions for the problem where each basis function has a large information content.

The reduced basis method





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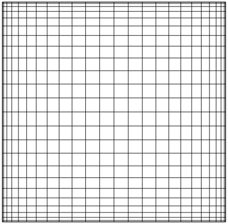

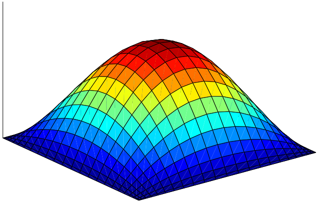

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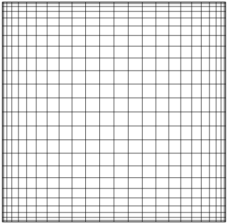

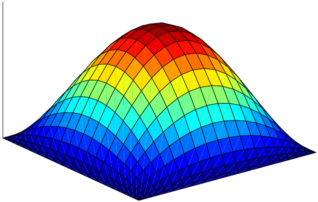

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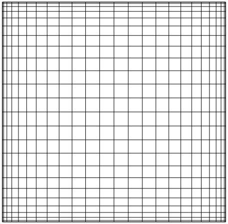

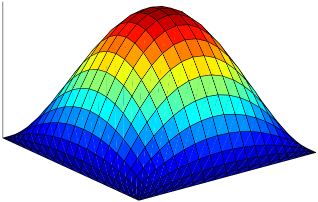

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The reduced basis method

Parameter dependent problems: $F(u;\mu) = 0$

Given $\mu \in \mathcal{D} \subset \mathbb{R}^p$, find $u \in X_{\mathcal{N}}$ such that

$$a(u, v; \mu) = \ell(v) \quad \forall\, v \in X_{\mathcal{N}}, \quad \mathcal{N} \gg 1$$

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Reduced basis procedure

Offline:

Basis functions defined on $X_{\mathcal{N}}$ span the reduced basis approximation space

$$X_N = span\{u_i\}_{i=1}^{N}, \quad N \ll \mathcal{N}$$

The reduced basis method

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$$X_N = span\{u_i\}_{i=1}^N, \quad N \ll \mathcal{N}$$

Online:

Find the reduced basis approximation:

$$u_N(\mu) = \sum_{i=1}^N \alpha_i(\mu) u_i,$$

such that

$$a(u_N, v; \mu) = \ell(v) \quad \forall\, v \in X_N$$

The reduced basis method

Offline/online decoupling

- GOAL: Avoid online computations on the underlying FEM basis.
- All basis functions in X_N are stored in the FEM basis, so both $a(u_N, v; \mu)$ and $\ell(v)$ involve computations on the FEM basis.
- $\ell(v)$ is independent of the parameter, so this can be done offline.

The reduced basis method

Offline/online decoupling

If the problem has affine parameter dependence, we may decouple the bilinear form such that

$$a(u, v; \mu) = \sum_{q=1}^{Q} \sigma^q(\mu) a^q(u, v).$$

The reduced basis method

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Offline we compute $A_{ij}^q = a^q(u_j, u_i),\ i, j = 1, ..., N,\ q = 1, ..., Q$, and $\ell_i = \ell(u_i),\ i = 1, ..., N$ using $\mathcal{O}(QN^2\mathcal{N})$ operations.

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Online we then solve the algebraic equations

$$(\sum_{q=1}^{Q} \sigma^q(\mu) A^q)\underline{\alpha} = \underline{\ell},$$

using $\mathcal{O}(QN^2)$ operations on assembly, and $\mathcal{O}(N^3)$ operations to find $\underline{\alpha}$.

The reduced basis method

Output of interest

The reduced basis approximation $u_N(\mu) = \sum_{i=1}^{N} \alpha_i(\mu) u_i$ needs $\mathcal{O}(N\mathcal{N})$ operations for assembly.

The reduced basis method

Output of interest

The reduced basis approximation $u_N(\mu) = \sum_{i=1}^{N} \alpha_i(\mu) u_i$ needs $\mathcal{O}(N\mathcal{N})$ operations for assembly.
A derived quantity (or output of interest) $s(u) = f(u(\mu))$ (e.g., drag force, volume flow rate,) may be computed as

$$s(u_N) = f(u_N(\mu)) = f(\sum_{i=1}^{N} \alpha_i(\mu) u_i) = \sum_{i=1}^{N} \alpha_i(\mu) f_i,$$

where $f_i = f(u_i)$ may be computed offline. We thus only need N operations to find $s(u_N)$ once $\underline{\alpha}$ is found.
Recall $N \ll \mathcal{N}$.

The reduced basis method

A posteriori error estimation

For a given output of interest $s(u)$, Prud'homme et al. (2002) present upper and lower bounds, such that

$$s^{-}(u_N) \leq s(u) \leq s^{+}(u_N)$$

For affine parameter dependence they also show that through offline/online decoupling of the computations, the online work needed to find the output bounds only depends on N.

The reduced basis method

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Adaptive reduced basis method

Based on the bound gap

$$\Delta s(u_N) = s^{+}(u_N) - s^{-}(u_N),$$

Veroy et al. (2003) present an adaptive method to control the number of basis functions N used in the reduced basis approximation.

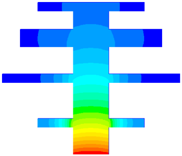
The reduced basis method

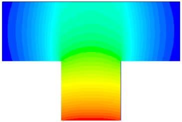
A greedy approach

Recall that $\mu \in \mathcal{D} \subset \mathbb{R}^p$, and let $\Xi_n \equiv \{\mu_1^*, ..., \mu_n^*\}$ be a finite dimensional substitute for $\mathcal{D}$.
For an initial subset $S_{N_0} = \{\mu_1, ..., \mu_{N_0}\} \subset \Xi_n$, and
$X_{N_0} = \text{span}\ \{u(\mu_1), ..., u(\mu_{N_0}\}$

$$
\begin{array}{l}
\text{for } N = N_0 + 1 : N_{\max} \\
\quad \mu_N = \arg\max_{\mu \in \Xi_n} \Delta s(u_{N-1}(\mu)) \\
\quad \text{if } \Delta s(u_{N-1}(\mu_N)) \leq \text{tol} \\
\quad\quad \text{exit} \\
\quad \text{end} \\
\quad S_N = S_{N-1} \cup \mu_N \\
\quad X_N = X_{N-1} + \text{span}\ \{u(\mu_N)\} \\
\text{end}
\end{array}
$$

The reduced basis element method

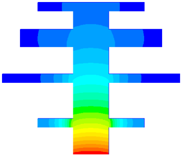


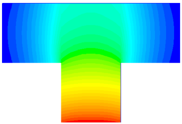


Main idea

Combine the reduced basis method with domain decomposition to solve PDEs in complex geometries

The reduced basis element method





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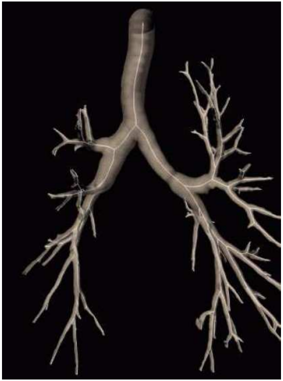
Applications

- Repetitive geometry: Thermal fin and Hierarchical systems
- Repetitive solves: Optimization and Control

The reduced basis element method



Marc Thiriet et al, INRIA

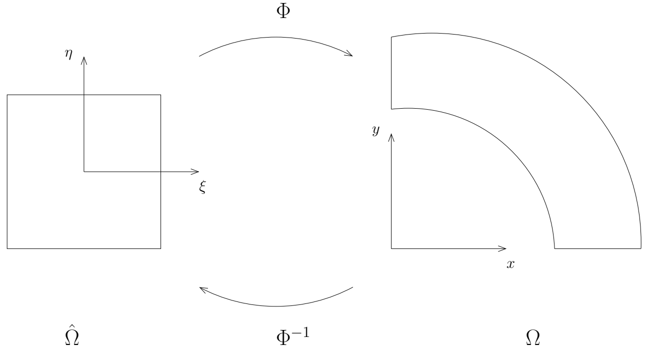
Building blocks

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}^k = \bigcup_{k=1}^{K} \overline{\Phi^k(\hat{\Lambda})}$$

Pipes $\{\Omega^k = \Phi^k(\hat{\Omega})\}_{k=1}^{K_P}$, Bifurcations $\{\Omega^k = \Phi^k(\hat{\mathcal{B}})\}_{k=K_P+1}^{K_P+K_B}$.

The reduced basis element method

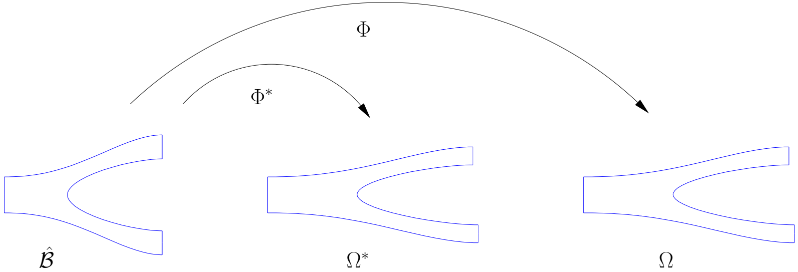
Building blocks: pipes



$$\Omega = \Phi(\hat{\Omega}), \quad \mathcal{J}(\Phi) = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}, \quad J(\Phi) = det(\mathcal{J}(\Phi)),$$

The reduced basis element method

Building blocks: bifurcations



$$\Omega = \Phi(\hat{\mathcal{B}}), \quad \mathcal{J}(\Phi) = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}, \quad J(\Phi) = det(\mathcal{J}(\Phi)),$$

The reduced basis element method

The geometry as a parameter

$$a(\mathbf{v}, \mathbf{w}; \Phi) = \nu \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{w} \, d\Omega, \quad \Omega = \Phi(\hat{\Lambda})$$

The reduced basis element method

The geometry as a parameter

$$a(\mathbf{v}, \mathbf{w}; \Phi) = \nu \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{w} \, d\Omega, \quad \Omega = \Phi(\hat{\Lambda})$$

$$\nabla = \mathcal{J}^{-T}(\Phi)\hat{\nabla}$$

The reduced basis element method

The geometry as a parameter

$$a(\mathbf{v}, \mathbf{w}; \Phi) = \nu \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \mathbf{w} d\Omega, \quad \Omega = \Phi(\hat{\Lambda})$$

$$\nabla = \mathcal{J}^{-T}(\Phi)\hat{\nabla}$$

$$a(\mathbf{v}, \mathbf{w}; \Phi) = \nu \int_{\hat{\Lambda}} \mathcal{J}^{-T}(\Phi)\hat{\nabla}(\mathbf{v} \circ \Phi) \cdot \mathcal{J}^{-T}(\Phi)\hat{\nabla}(\mathbf{w} \circ \Phi) |J(\Phi)| \, d\hat{\Lambda}.$$

The reduced basis element method

The steady Stokes equations: weak form

Let $\Omega = \Phi(\hat{\Lambda})$. Find $\mathbf{u} \in X(\Omega)$ and $p \in M(\Omega)$, such that

$$\begin{array}{rcll} a(\mathbf{u}, \mathbf{v}; \Phi) + b(\mathbf{v}, p; \Phi) & = & \ell(\mathbf{v}; \Phi) & \forall\, \mathbf{v} \in X(\Omega) \\ b(\mathbf{u}, q; \Phi) & = & 0 & \forall\, q \in M(\Omega) \end{array}$$

$$\begin{array}{rcl} X(\Omega) & = & \{\mathbf{v} \in (H^1(\Omega))^2, \mathbf{v}^w = 0, v_t^{in} = v_t^{out} = 0\} \\ M(\Omega) & = & L^2(\Omega) \end{array}$$

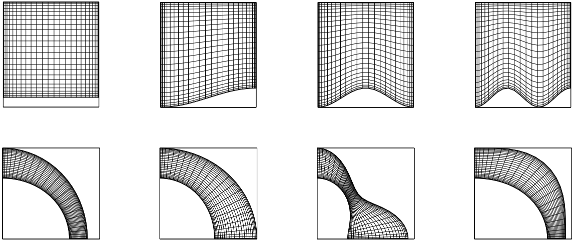
$$a(\mathbf{v}, \mathbf{w}) = \nu \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{w} d\Omega$$

$$b(\mathbf{v}, q) = -\int_\Omega q \nabla \cdot \mathbf{v} d\Omega$$

Neumann type boundary conditions by specifying $\sigma_n = \frac{\partial u_n}{\partial n} - p$ to be $\sigma_n^{in} = -1$ along Γ_{in}, and $\sigma_n^{out} = 0$ along Γ_{out}.

The reduced basis element method

Preselected geometries



The reduced basis spaces

The parameter space $S_N = \{\Phi_i\}_{i=1}^{N}$.

The reduced basis element method

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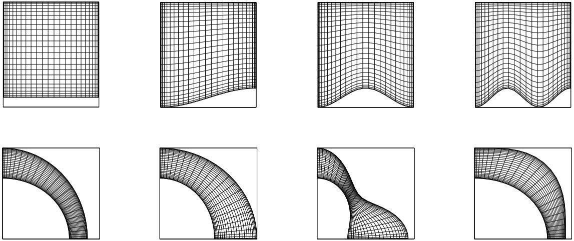

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The velocity space $\hat{X}_N^0 = span\{\hat{\mathbf{u}}_i = \Psi_i(\mathbf{u}_i) = \mathcal{J}_i^{-1}(\mathbf{u}_i \circ \Phi_i)|J_i|\}_{i=1}^N$.

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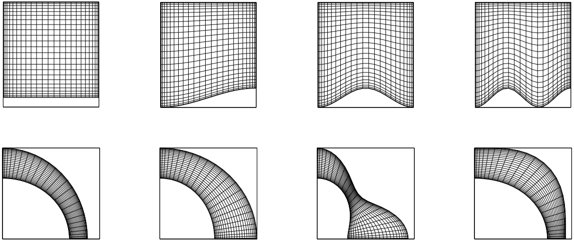

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The pressure space $\hat{M}_N = span\{\hat{p}_i = p_i \circ \Phi_i\}_{i=1}^N$.

The reduced basis element method

Reduced basis steady Stokes approximation

Find $\mathbf{u}_N \in X_N(\Omega)$ and $p_N \in M_N(\Omega)$, such that

$$\begin{array}{lcll} a(\mathbf{u}_N, \mathbf{v}; \Phi) + b(\mathbf{v}, p_N; \Phi) & = & \ell(\mathbf{v}; \Phi) & \forall\, \mathbf{v} \in X_N(\Omega) \\ b(\mathbf{u}_N, q; \Phi) & = & 0 & \forall\, q \in M_N(\Omega) \end{array}$$

The reduced basis element method

Reduced basis steady Stokes approximation

Find $\mathbf{u}_N \in X_N^0(\Omega)$, such that

$$a(\mathbf{u}_N, \mathbf{v}; \Phi) = \ell(\mathbf{v}; \Phi) \qquad \forall\, \mathbf{v} \in X_N^0(\Omega)$$

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Enriched reduced basis velocity space

For each $\hat{p}_i \in \hat{M}_N$ we find $\hat{\mathbf{v}}_i \in \hat{X}_{\mathcal{N}} = X_{\mathcal{N}}(\hat{\Lambda})$ such that

$$\hat{\mathbf{v}}_i = \arg\max_{\hat{\mathbf{u}} \in \hat{X}_{\mathcal{N}}} \frac{\int_{\hat{\Lambda}} \hat{p}_i \hat{\nabla} \cdot \hat{\mathbf{u}}\, d\hat{\Lambda}}{|\hat{\mathbf{u}}|_{H^1}}.$$

The enriched reduced basis velocity space on Ω is then

$$X_N(\Omega) = \{\mathbf{v} \in X_N^0(\Omega) \oplus X_N^e(\Omega)\},$$

where $X_N^0(\Omega) = \{\Psi^{-1}(\hat{\mathbf{u}}_i)\}_{i=1}^N$, and $X_N^e(\Omega) = \{\Psi^{-1}(\hat{\mathbf{v}}_i)\}_{i=1}^N$.

The reduced basis element method

Compliant output of interest

Volume flow rate: $s(\mathbf{u}) = \ell(\mathbf{u}; \Phi)$

Compute output bounds: $s^-(\mathbf{u}_N) \leq s(\mathbf{u}) \leq s^+(\mathbf{u}_N)$

The output bounds for steady Stokes flow

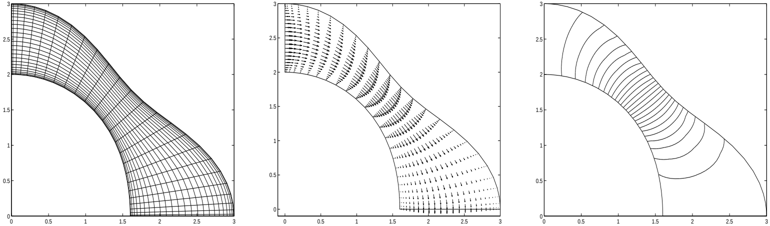
$$\begin{aligned} s^-(\mathbf{u}_N) &= \ell(\mathbf{u}_N; \Phi) \\ s^+(\mathbf{u}_N) &= \ell(\mathbf{u}_N; \Phi) + \hat{a}(\mathbf{e}, \mathbf{e}; \Phi) \end{aligned}$$

$$\hat{a}(\mathbf{v}, \mathbf{w}; \Phi) = \int_{\hat{\Lambda}} q(\Phi) \hat{\nabla}(\mathbf{v} \circ \Phi) \cdot \hat{\nabla}(\mathbf{w} \circ \Phi) d\hat{\Lambda}$$

The reconstructed error $\mathbf{e}$ satisfies the residual equation

$$\hat{a}(\mathbf{e}, \mathbf{v}; \Phi) = \ell(\mathbf{v}; \Phi) - a(\mathbf{u}_N, \mathbf{v}; \Phi) - b(\mathbf{v}, p_N; \Phi) \quad \forall\, \mathbf{v} \in \tilde{X}_{\mathcal{N}}(\Omega)$$

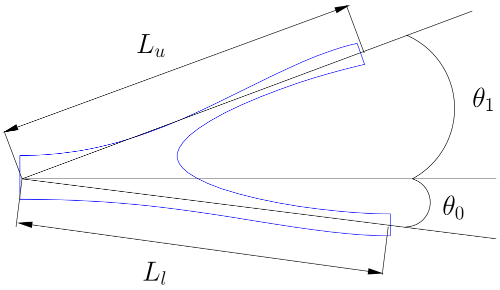
The reduced basis element method



The reduced basis steady Stokes error on a pipe

N	$\lvert \mathbf{u}_N - \mathbf{u}_{\mathcal{N}} \rvert_{H^1}$	$\lvert\lvert p_N - p_{\mathcal{N}} \rvert\rvert_{L^2}$	$s(\mathbf{u}_{\mathcal{N}}) - s^-(\mathbf{u}_N)$	$s^+(\mathbf{u}_N) - s(\mathbf{u}_{\mathcal{N}})$
3	$1.2 \cdot 10^{-2}$	$1.6 \cdot 10^{-1}$	$1.4 \cdot 10^{-4}$	$9.7 \cdot 10^{-2}$
6	$8.6 \cdot 10^{-3}$	$5.2 \cdot 10^{-2}$	$7.4 \cdot 10^{-5}$	$6.9 \cdot 10^{-3}$
9	$2.6 \cdot 10^{-3}$	$1.4 \cdot 10^{-2}$	$7.0 \cdot 10^{-6}$	$1.6 \cdot 10^{-3}$
12	$1.8 \cdot 10^{-3}$	$9.5 \cdot 10^{-3}$	$3.2 \cdot 10^{-6}$	$6.2 \cdot 10^{-4}$
15	$1.4 \cdot 10^{-3}$	$8.6 \cdot 10^{-3}$	$1.9 \cdot 10^{-6}$	$3.8 \cdot 10^{-4}$

The reduced basis element method



The reduced basis steady Stokes error on a bifurcation

N	$\lvert \mathbf{u}_N - \mathbf{u}_{\mathcal{N}} \rvert_{H^1}$	$\lvert\lvert p_N - p_{\mathcal{N}} \rvert\rvert_{L^2}$	$s(\mathbf{u}_{\mathcal{N}}) - s^-(\mathbf{u}_N)$	$s^+(\mathbf{u}_N) - s(\mathbf{u}_{\mathcal{N}})$
1	$1.4 \cdot 10^{-2}$	$8.8 \cdot 10^{-2}$	$2.1 \cdot 10^{-4}$	$1.9 \cdot 10^{-3}$
5	$5.0 \cdot 10^{-4}$	$4.8 \cdot 10^{-3}$	$2.5 \cdot 10^{-7}$	$1.1 \cdot 10^{-5}$
10	$9.9 \cdot 10^{-6}$	$7.2 \cdot 10^{-5}$	$9.8 \cdot 10^{-11}$	$7.3 \cdot 10^{-9}$
15	$4.0 \cdot 10^{-6}$	$7.3 \cdot 10^{-6}$	$1.6 \cdot 10^{-11}$	$1.5 \cdot 10^{-11}$

Offline/online decoupling: Empirical interpolation

Offline/online decoupling: Empirical interpolation

Affine parameter dependence

$$a(\mathbf{v}, \mathbf{w}; \mu) = \sum_{q=1}^{Q} \sigma^q(\mu) a^q(\mathbf{v}, \mathbf{w})$$

Offline: Compute $A^q_{ij} = a^q(\mathbf{u}_i, \mathbf{u}_j)$ for $q = 1, ..., Q$ and $i, j = 1, ..., N$.
Online: Assemble A in $\mathcal{O}(QN^2)$ operations, and solve the reduced problem in $\mathcal{O}(N^3)$ operations.

Offline/online decoupling: Empirical interpolation

Non-affine parameter dependence

$$a(\mathbf{v}, \mathbf{w}; \Phi) = \nu \int_{\hat{\Lambda}} \mathcal{J}^{-T}(\Phi)\hat{\nabla}(\mathbf{v} \circ \Phi) \cdot \mathcal{J}^{-T}(\Phi)\hat{\nabla}(\mathbf{w} \circ \Phi)|J(\Phi)|\, d\hat{\Lambda}.$$

Offline/online decoupling: Empirical interpolation

Non-affine parameter dependence

$$a(\mathbf{v}, \mathbf{w}; \Phi) = \nu \int_{\hat{\Lambda}} \mathcal{J}^{-T}(\Phi)\hat{\nabla}(\mathbf{v} \circ \Phi) \cdot \mathcal{J}^{-T}(\Phi)\hat{\nabla}(\mathbf{w} \circ \Phi)|J(\Phi)| \, d\hat{\Lambda}.$$

$$a(\mathbf{v}, \mathbf{w}; \Phi) = \nu \sum_{q=1}^{Q} \int_{\hat{\Lambda}} g^q(\Phi) a^q(\hat{\mathbf{v}}, \hat{\mathbf{w}}) \, d\hat{\Lambda}$$

Offline/online decoupling: Empirical interpolation

Non-affine parameter dependence

$$a(\mathbf{v}, \mathbf{w}; \Phi) = \nu \int_{\hat{\Lambda}} \mathcal{J}^{-T}(\Phi) \hat{\nabla}(\mathbf{v} \circ \Phi) \cdot \mathcal{J}^{-T}(\Phi) \hat{\nabla}(\mathbf{w} \circ \Phi) |J(\Phi)| \, d\hat{\Lambda}.$$

$$a(\mathbf{v}, \mathbf{w}; \Phi) = \nu \sum_{q=1}^{Q} \int_{\hat{\Lambda}} g^q(\Phi) a^q(\hat{\mathbf{v}}, \hat{\mathbf{w}}) \, d\hat{\Lambda}$$

$$a(\mathbf{v}, \mathbf{w}; \Phi) \approx \nu \sum_{q=1}^{Q} \sum_{m=1}^{M} \beta_m^q(\Phi) \int_{\hat{\Lambda}} \tilde{g}^q(\Phi_m) a^q(\hat{\mathbf{v}}, \hat{\mathbf{w}}) \, d\hat{\Lambda}$$

Offline: compute $A_{ij}^{qm} = \tilde{g}^q(\Phi_m) a^q(\mathbf{u}_i, \mathbf{u}_j)$ for $q = 1, \dots, Q$, $m = 1, \dots, M$, and $i, j = 1, \dots, N$.
Online: Find $\beta_m^q(\Phi)$, assemble A in $\mathcal{O}(QMN^2)$ operations, and solve the reduced problem in $\mathcal{O}(N^3)$ operations

Offline/online decoupling: Empirical interpolation

Empirical interpolation

Find the constants $\beta_m^q(\Phi)$, $m = 1, ..., M$ such that.

$$\sum_{m=1}^{M} \beta_m^q(\Phi) g^q(\Phi_m) \approx g^q(\Phi).$$

Projection gives

$$[b_{mn}]\,[\beta_n] = [g_m]\,,$$

where $b_{mn} = \int_{\hat{\Lambda}} g^q(\Phi_m) g^q(\Phi_n) d\hat{\Lambda}$
and $g_m = \int_{\hat{\Lambda}} g^q(\Phi) g^q(\Phi_m) d\hat{\Lambda}$.

Offline/online decoupling: Empirical interpolation

Empirical interpolation

Barrault et al. (2004) introduced magic points $\{t_m^q \in \hat{\Lambda}\}_{m=1}^M$ and corresponding operators $\{\tilde{g}^q(\Phi_m(x))\}_{m=1}^M$ such that

$$
\begin{aligned}
\tilde{g}^q(\Phi_m(t_m^q)) &= 1, \qquad m = 1, \ldots, M \\
\tilde{g}^q(\Phi_m(t_n^q)) &= 0, \qquad m < n.
\end{aligned}
$$

The matrices B^q defined by $B_{mn}^q = \tilde{g}^q(\Phi_m(t_n^q))$ are lower triangular.

Offline/online decoupling: Empirical interpolation

Empirical interpolation

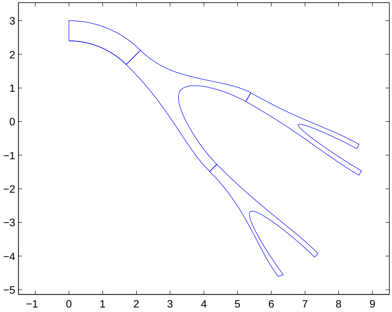
Online we only need to find the coefficients $\beta_m^q(\Phi)$ to compute

$$a(\mathbf{u}, \mathbf{v}; \Phi) \approx \nu \sum_{q_1}^{Q} \sum_{m=1}^{M} \beta_m^q(\Phi) \int_{\hat{\Omega}} \tilde{g}^q(\Phi_m) a^q(\hat{\mathbf{u}}, \hat{\mathbf{v}}) \, d\hat{\Omega}$$

The coefficients are found by solving

$$\sum_{n=1}^{M} B_{mn}^q \beta_n^q(\Phi) = g(\Phi(t_m^q)), \quad 1 \leq m \leq M$$

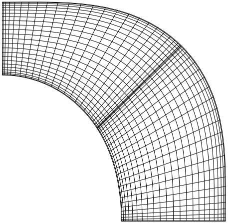
Multi-block domains



New features

- Boundary conditions
- Continuity across block interfaces
- Nonconforming reduced basis element solution

Multi-block domains



Boundary conditions

- Construct one set of basis functions, and use the reflection across the vertical axis on the reference domain to deal with symmetric boundary conditions.
- Construct several sets of basis functions, each set corresponding to different boundary conditions.

Multi-block domains

Constraints ("gluing")

Let $\overline{\Gamma}_{kl} = \overline{\mathcal{B}}^k \bigcap \overline{\mathcal{B}}^l$

$$\int_{\Gamma_{kl}} (\mathbf{v}^k - \mathbf{v}^l) \cdot \mathbf{n}\psi \, ds = 0, \quad \forall \, \psi \in W^n_{k,l}, \ \ \forall k, l, \tag{1}$$

$$\int_{\Gamma_{kl}} (\mathbf{v}^k - \mathbf{v}^l) \cdot \mathbf{t}\psi \, ds = 0, \quad \forall \, \psi \in W^t_{k,l}, \ \ \forall k, l, \tag{2}$$

Multi-block domains

Constraints ("gluing")

Let $\overline{\Gamma}_{kl} = \overline{\mathcal{B}}^k \bigcap \overline{\mathcal{B}}^l$

$$\int_{\Gamma_{kl}} (\mathbf{v}^k - \mathbf{v}^l) \cdot \mathbf{n}\psi \; ds = 0, \quad \forall \; \psi \in W^n_{k,l}, \;\; \forall k, l, \tag{1}$$

$$\int_{\Gamma_{kl}} (\mathbf{v}^k - \mathbf{v}^l) \cdot \mathbf{t}\psi \; ds = 0, \quad \forall \; \psi \in W^t_{k,l}, \;\; \forall k, l, \tag{2}$$

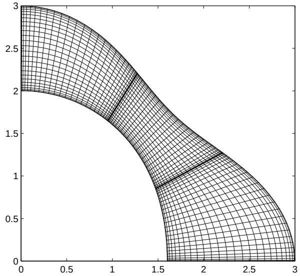
Reduced basis approximation (nonconforming)

Find $\mathbf{u}_N \in X_N(\Omega)$ and $p_N \in M_N(\Omega)$, such that

$$\begin{array}{lcll} a(\mathbf{u}_N, \mathbf{v}; \Phi) + b(\mathbf{v}, p_N; \Phi) & = & \ell(\mathbf{v}; \Phi) & \forall \; \mathbf{v} \in X_N(\Omega) \\ b(\mathbf{u}_N, q; \Phi) & = & 0 & \forall \; q \in M_N(\Omega) \end{array}$$

where $X_N(\Omega) = \{\mathbf{v} \in X^0_N(\Omega) \oplus X^e_N(\Omega), \quad \text{s.t.}(1) \text{ and } (2) \text{ hold } \}$.

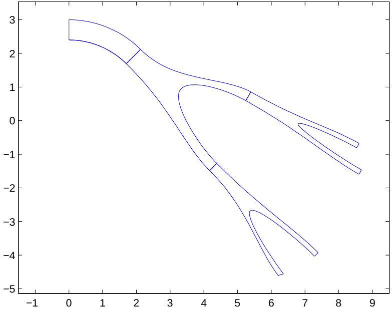
Example I: Three pipe segments



The reduced basis error

N	N_1	$\lvert \mathbf{u}_N - \mathbf{u}_{\mathcal{N}} \rvert_{H^1}$	$\lvert\lvert p_N - p_{\mathcal{N}} \rvert\rvert_{L^2}$
27	9	$2.3 \cdot 10^{-3}$	$3.6 \cdot 10^{-1}$
33	11	$1.2 \cdot 10^{-3}$	$5.8 \cdot 10^{-2}$
39	13	$9.7 \cdot 10^{-4}$	$4.4 \cdot 10^{-3}$
45	15	$8.4 \cdot 10^{-4}$	$3.6 \cdot 10^{-3}$

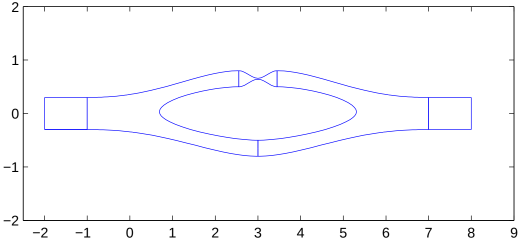
Example II: Hierarchical flow system



Reduced basis error

N	N_1	N_2	$\lvert \mathbf{u}_N - \mathbf{u}_{\mathcal{N}} \rvert_{H^1}$	$\lvert\lvert p_N - p_{\mathcal{N}} \rvert\rvert_{L^2}$
36	9	9	$2.6 \cdot 10^{-3}$	$4.0 \cdot 10^{-1}$
44	11	11	$1.7 \cdot 10^{-3}$	$6.6 \cdot 10^{-2}$
52	13	13	$1.2 \cdot 10^{-3}$	$4.9 \cdot 10^{-2}$
65	15	15	$1.1 \cdot 10^{-3}$	$3.7 \cdot 10^{-2}$
105	15	30	$4.2 \cdot 10^{-4}$	$6.3 \cdot 10^{-3}$

Example III: A "bypass"



Reduced basis error

N	N_1	N_2	$\lvert \mathbf{u}_N - \mathbf{u}_{\mathcal{N}} \rvert_{H^1}$	$\lvert\lvert p_N - p_{\mathcal{N}} \rvert\rvert_{L^2}$
45	9	9	$9.3 \cdot 10^{-3}$	$3.3 \cdot 10$
55	11	11	$3.1 \cdot 10^{-3}$	$5.3 \cdot 10^{-1}$
65	13	13	$2.3 \cdot 10^{-3}$	$9.0 \cdot 10^{-2}$
75	15	15	$1.4 \cdot 10^{-3}$	$5.3 \cdot 10^{-2}$
105	15	30	$5.4 \cdot 10^{-4}$	$3.0 \cdot 10^{-2}$

Reduced basis modeling of complex flow systems

Key features

- Geometry as a parameter
- Output bounds
- Offline/online decoupling
- Building blocks
- Lagrange multipliers

Reduced basis modeling of complex flow systems

Key features

- Geometry as a parameter
- Output bounds
- Offline/online decoupling
- Building blocks
- Lagrange multipliers

Future work/ongoing work

- *A posteriori* bounds on multi-block geometries
- Three dimensional domains
- Time dependent problems
- Fluid-structure interaction

Thank you!