

Finite Element Methods in General Relativity

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Some science history

- Newton's "*Principia*" (1687); Maxwell's "*Theory*" (1864)
- Newtonian gravity:
 - Gravitational potential for a point mass m : $\Phi = \frac{-Gm}{r}$
 - Corresponding acceleration: $\mathbf{g} = -\nabla\Phi = \frac{-Gm}{r^2}\mathbf{e}_r$
- Unable to explain:
 - Bending of light due to stars
 - Magnitude of the precession of the orbit of Mercury
 - $5600 - 5557 = 43$ seconds of arc per century
- Action at a distance

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Warming up with special relativity

- Maxwell's equations $\Rightarrow$ Speed of EM waves $c \simeq 3 \times 10^8$ m/s
... with respect to what?
- *"The laws of physics are the same in all uniformly moving reference frames"*
- Interesting implications:
 - Cosmic speed limit: $1 \leq \gamma = \frac{1}{\sqrt{1-v^2/c^2}} < \infty$
 - Time dilation: $\Delta t' = \gamma \Delta t$
 - Lorentz contraction: $\Delta x' = \frac{\Delta x}{\gamma}$
 - Mass-energy equivalence: $E = Mc^2 = \gamma mc^2$
- Minkowski metric: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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“Do not worry about your difficulties in mathematics, I assure you that mine are greater.” — Einstein

A crash course on modern geometry and topology

- **Spacetime:** Curved, pseudo-Riemannian manifold with a metric of signature $(-+++)$ $\Rightarrow$ Charts and atlases allow us to relate them to Euclidean spaces, $\mathbb{R}^n$
- **Tensor:** Multi-index object which transforms according to
$$\hat{A}_{j_1 \dots j_p}^{i_1 \dots i_q} = X_{k_1}^{i_1} \dots X_{k_q}^{i_q} Y_{j_1}^{l_1} \dots Y_{j_p}^{l_p} A_{l_1 \dots l_p}^{k_1 \dots k_q}$$
- **Metric:** Evolving, non-flat, symmetric, 2-index tensor, $g_{\mu\nu}$
- **Christoffel symbols:** $\Gamma_{jks} = \frac{1}{2} \left(\frac{\partial g_{js}}{\partial w^k} + \frac{\partial g_{ks}}{\partial w^j} - \frac{\partial g_{jk}}{\partial w^s} \right)$
- **Covariant derivative:** $Y^i_{;j} = Y^i_{,j} + \Gamma_{jk}^i Y^k$
- **Riemann curvature tensor:**
$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}$$
- **Ricci tensor:** $R_{ij} = R^k_{ikj}$
- **Scalar curvature:** $R = R^i_i$

A final prelude to general relativity

- What about action at a distance?
- What is so special about special relativity?
 - ... Physics is the same for all observers *in uniform motion*
- Do you know if you are in inertial reference frame?

Impossible to tell! $\Rightarrow$ Principle of equivalence

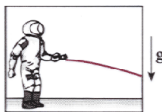
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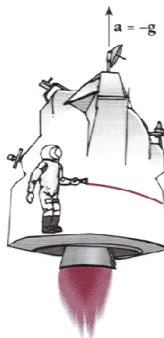
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Laboratory in
gravitational field

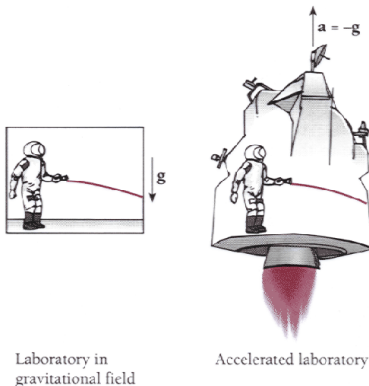


Accelerated laboratory

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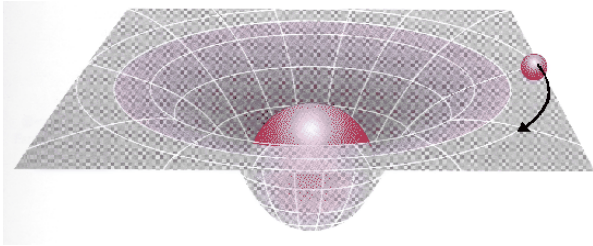
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A new basis for gravity



Gravity is the geometry of spacetime!

A look at the field equations

- System of second order, coupled, nonlinear PDEs:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

G — Gravitational constant

c — Velocity of light

- **Einstein Tensor:** $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$

$R_{\mu\nu}$ — Ricci tensor

R — Scalar curvature

- **Stress energy tensor:** $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$

Assuming a perfect fluid with 4-velocity U^μ , for e.g.

- Covariant divergence of $\mathbf{G}$ and $\mathbf{T} = 0 \Rightarrow$ Conservation laws

A famous analytical solution

- Working in a coordinate chart with (r, θ, ϕ, t)

- Spherically symmetric, static spacetime

- General form of such a metric:

$$ds^2 = A(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 + B(r) dt^2$$

- Vacuum field equations: $R_{ab} = 0 \Rightarrow$

$$4\dot{A}\dot{B}^2 - 2r\ddot{B}AB + r\dot{A}\dot{B}\dot{B} + r\dot{B}^2 A = 0$$

$$r\dot{A}\dot{B} + 2A^2\dot{B} - 2AB - r\dot{B}A = 0$$

$$-2r\ddot{B}AB + r\dot{A}\dot{B}\dot{B} + r\dot{B}^2 A - 4\dot{B}AB = 0$$

Unique solution:

$$ds^2 = \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) - c^2 \left(1 - \frac{2Gm}{c^2 r}\right) dt^2$$

using the weak field approximation: $g_{00} = -c^2 + \frac{2Gm}{r}$

Some ado about numerics

- Formulations in *weak form* exist for “ $3 + 1$ ” space $\times$ time decomposition (for FEM)
(First we develop an elegant covariant theory and then turn it back into a $3 + 1$ form!)
- A typical numerical scheme
 - Slice spacetime into spacelike 3D hyperspaces; Successive slices are like “instants” of time
 - Use the constraint equations and solve for the conditions on the initial hypersurface
 - Evolve these solutions forward
 - Periodically check if constraints are propagated correctly
- FeTK: Open source finite element software libraries for solving coupled PDEs on manifolds

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The usefulness of it all

- Better understanding of the physics of our universe
 - Calculates precession of mercury's orbit correctly!
- Simulations for gravitational wave detectors
 - Recall this is a field theory, no action at a distance
- Physics of black holes
 - Accretion disk evolution around black holes
 - Jet formation near black holes
- Relativistic flows: Jets, Shocks
- . . .

An incomplete bibliography

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○ <http://psi.phys.wits.ac.za/teaching/Connell/>