Toward a goal-oriented error-controlled solver for the incompressible Navier-Stokes equations

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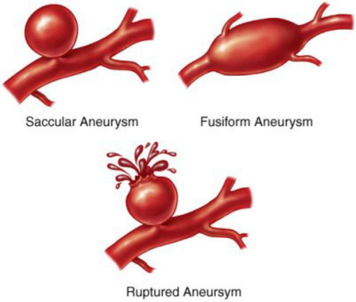
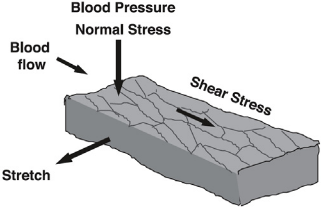
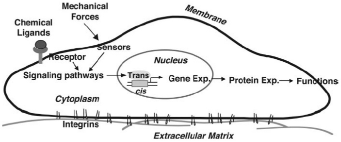
A brief look at the biology



How aneurysms look



Clarifying different directions



A schematic of the endothelial cell

- Wall shear stress-driven apoptotic behavior of muscle cells
- Remodeling of the arterial wall under constant tension

Quantities we are consequently interested in

- *Fluid shear stress*: Product of fluid viscosity and the velocity gradient between adjacent layers (fluid mechanics)
- *Stretch:* Circumferential stress acts along the vessel wall perimeter to cause stretching (fluid, solid mechanics)

[Show representative branched mesh]

The Navier-Stokes equations

Strong form:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\boldsymbol{\nabla} p}{\rho} - \nu \nabla^2 \boldsymbol{u} + (\boldsymbol{\nabla} \boldsymbol{u})\, \boldsymbol{u} = \boldsymbol{f}\ ;\ \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

An (approximate) weak form:

Find $(\boldsymbol{u}^{k+1}, p^{k+1}) \in V_u \times V_p$ such that

$$\left(\frac{D\boldsymbol{u}^{k+1}}{\Delta t}, \boldsymbol{v}\right) + \nu(\boldsymbol{\nabla} \boldsymbol{u}^{k+1}, \boldsymbol{\nabla} \boldsymbol{v}) - \left(\frac{p^{\bigstar,k+1}}{\rho}, \nabla \cdot \boldsymbol{v}\right) = (\boldsymbol{g}^{k+1}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \hat{V}$$

$$\text{where} \quad \boldsymbol{g}^{k+1} = \boldsymbol{f}^{k+1} - ((\boldsymbol{\nabla} \boldsymbol{u})\, \boldsymbol{u})^{\bigstar,k+1}$$

$$(\nabla \phi, \frac{\nabla \psi^{k+1}}{\rho}) = (\nabla \phi, \frac{D\boldsymbol{u}^{k+1}}{\Delta t}) \quad \forall \phi \in \hat{V}_\phi$$

$$(\frac{p^{k+1}}{\rho}, q) = (\frac{p^{\bigstar,k+1}}{\rho} + \frac{\psi^{k+1}}{\rho} - \nu \nabla \cdot \boldsymbol{u}^{k+1}, q) \quad \forall q \in \hat{V}_p$$

[Guermond and Shen, 2003]

A first step: Stokes equations and today's goal functionals

Steady state in strong form:

$$\operatorname{div}\left(\boldsymbol{\sigma}(\boldsymbol{u},p)\right) + \boldsymbol{f} = \mathbf{0};\ \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$
$$\boldsymbol{\sigma}(\boldsymbol{u},p) = 2\ \mu\ \mathrm{grad}_{\mathrm{sym}}(\boldsymbol{u}) - p\ \mathbf{1}$$

Goal functional 1 (Shear component of the traction):

$$S_{\mathrm{t}} = \int_{\Gamma} \left(\boldsymbol{\sigma}(\boldsymbol{u},p)\ \boldsymbol{n}\right)\cdot\boldsymbol{t}\,ds$$

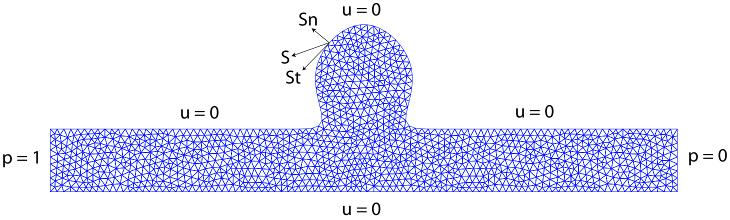
Goal functional 2 (Normal component of the traction):

$$S_{\mathrm{n}} = \int_{\Gamma} \left(\boldsymbol{\sigma}(\boldsymbol{u},p)\ \boldsymbol{n}\right)\cdot\boldsymbol{n}\,ds$$

Error indicator:

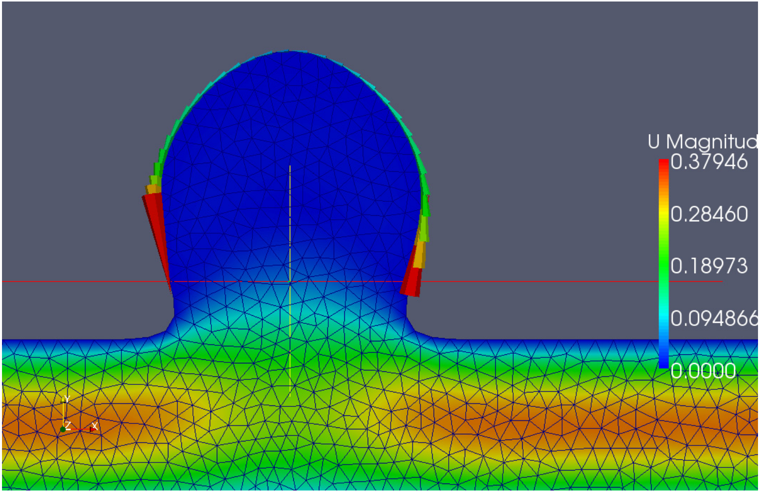
$$S_{\mathrm{n,t}}\left(\boldsymbol{\sigma}(\boldsymbol{u},p)\right) - S_{\mathrm{n,t}}\left(\boldsymbol{\sigma}(\boldsymbol{u}^h,p^h)\right) \le \sum_K C_1\ h_K\ ||D\boldsymbol{w}||_K\ ||\underbrace{\operatorname{div}\left(\boldsymbol{\sigma}(\boldsymbol{u}^h,p^h) + f\right.}_{R_1}$$
$$+\sum_K C_2\ h_K\ ||Dr||_K\ ||\underbrace{\operatorname{div}(\boldsymbol{u}^h)}_{R_2}||_K$$
$$+\sum_K C_3\ \sqrt{h_K}\ ||D\boldsymbol{w}||_{\omega_K}\ ||[\partial_n \boldsymbol{u}_h]||_{\partial K}$$
$$+\sum_K C_4\ \sqrt{h_K}\ ||D\boldsymbol{w}||_{\omega_K}\ ||[p_h \boldsymbol{n}]||_{\partial K}$$

A representative problem in 2D



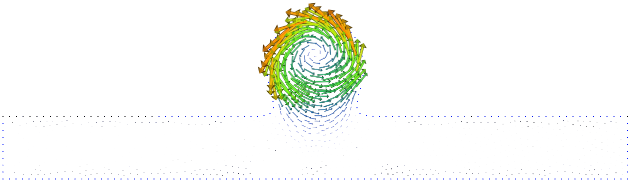
Initial mesh and boundary conditions

A representative problem in 2D



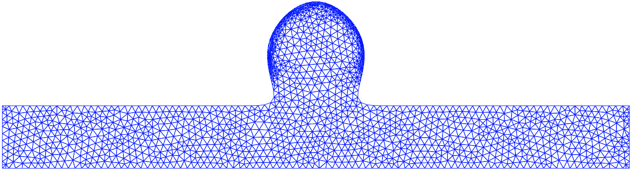
Colour contours of flow velocity magnitude and shear stresses

A representative problem in 2D



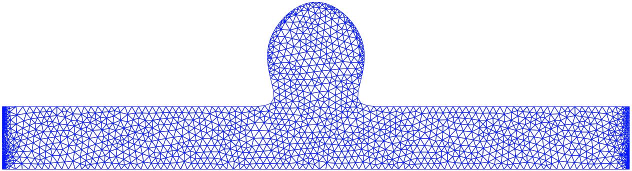
The dual velocity field driven by the shear force

Some results: Optimising for the shear component



After refining 5% of the cells with the highest error-indicators 10 times

Some results: Optimising for the normal component



After refining 5% of the cells with the highest error-indicators 10 times

What has been done, and what remains

- Determined error estimates for the Stokes problem
- Computed some components of these estimates
- Initial work on error estimates for a space-time finite element formulation for the Navier-Stokes equations
- Some work on the solid mechanics of the walls

- Computing remaining components of the error indicators
- Move to 3D: Entire implementation is ca. 200 lines of human readable Python which (in theory) trivially extends to 3D
- Incorporating non-linearity and time-dependency arising from the Navier-Stokes equations
- Couple solid mechanics to this computation in order to determine wall stretches

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