The spectral element method used to assess the quality of a deformed mesh

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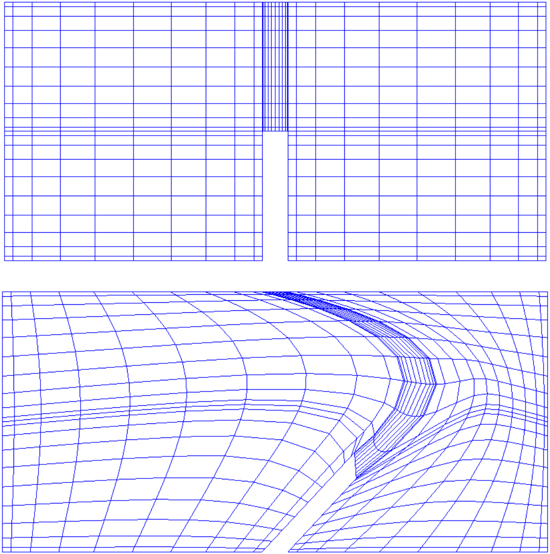
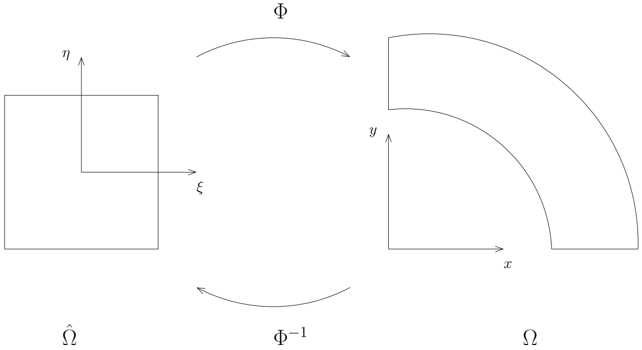
Introduction



Introduction



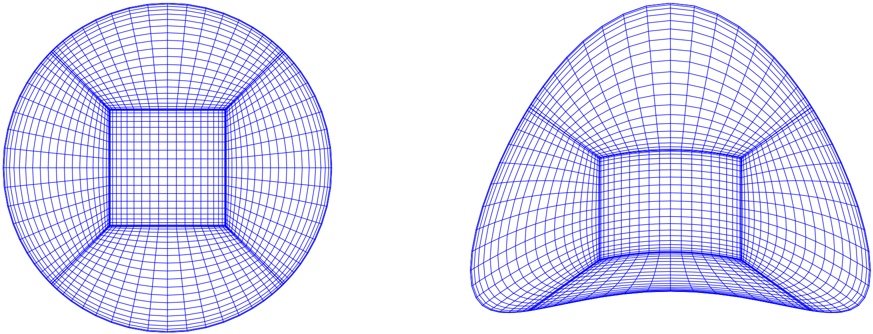
The Jacobian $\mathcal{J}$ of the map Φ is used to map velocity fields using the Piola transformation $\hat{\mathbf{u}}_i = \mathcal{J}_i^{-1}(\mathbf{u}_i \circ \Phi_i)|J_i|$. For a continuous velocity after transformation, the mapping Φ must be C^1.

Outline:

- Introduction
- Harmonic extension
- Barycentric extension
- Transfinite extension
- Generalized transfinite extension
- Regularity
- Results

Harmonic extension

$$
\begin{aligned}
\Delta \mathbf{x}(\xi,\eta) &= 0 && \text{in } \widehat{\Omega} \\
\mathbf{x}(\xi,\eta) &= \mathbf{g} && \text{on } \partial\widehat{\Omega}
\end{aligned}
$$



Harmonic extension

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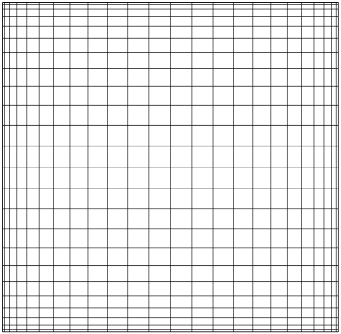
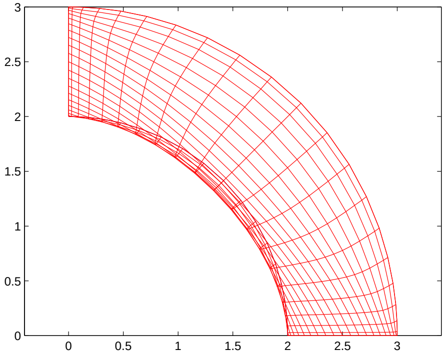




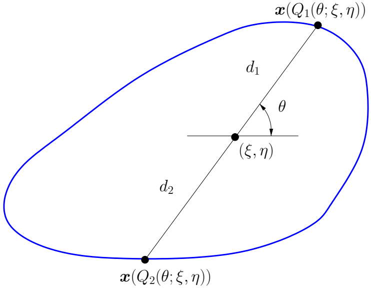
Barycentric extension

Pseudo-harmonic extension

$$\mathbf{x}(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{d_2(\theta)}{d_1(\theta)+d_2(\theta)}\mathbf{x}(Q_1(\theta)) + \frac{d_1(\theta)}{d_1(\theta)+d_2(\theta)}\mathbf{x}(Q_2(\theta))\right] d\theta$$

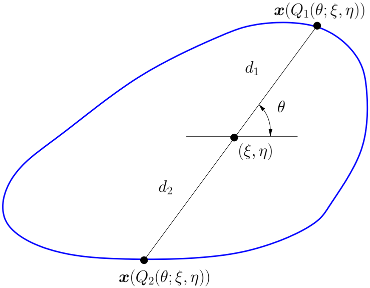
Mean-value interpolation

$$\mathbf{x}(\xi,\eta) = \int_0^{2\pi} \frac{1}{d_1(\theta)}\mathbf{x}(Q_1(\theta))d\theta \bigg/ \int_0^{2\pi} \frac{1}{d_1(\theta)} d\theta$$

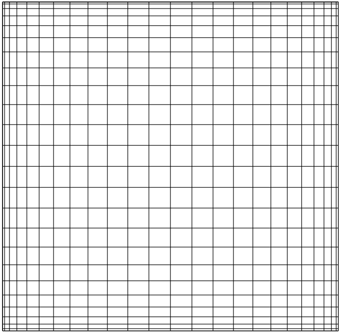


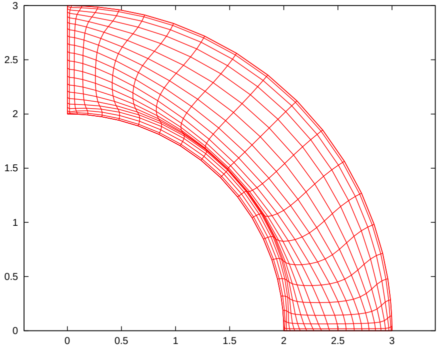
Barycentric extension

- On general domains the intersection points $Q_1(\theta;\xi,\eta)$ and $Q_2(\theta;\xi,\eta)$ requires the solution of a non-linear problem, but for a given reference domain they only need to be ocmputed once.
- If the reference domain is a circle, or a square, the intersection points can be given explicitly.
- Hermite versions are available, allowing larger deformations.

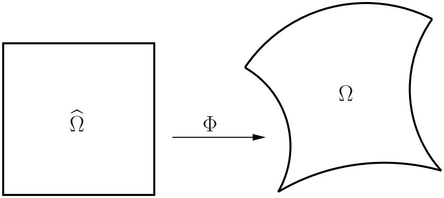


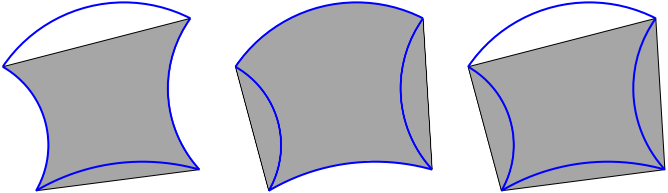
Hermite barycentric extension





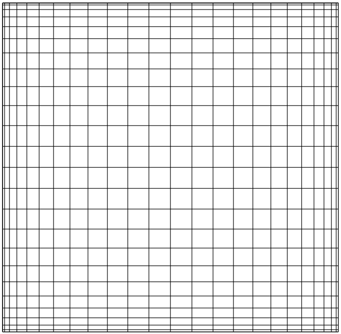
Transfinite extension (a.k.a. the Gordon-Hall algorithm)

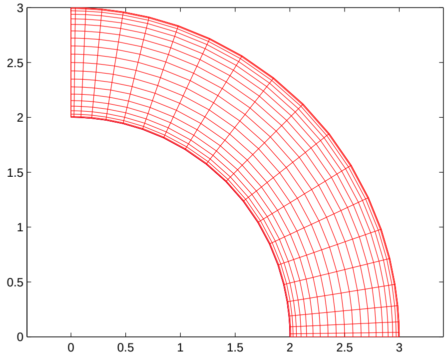




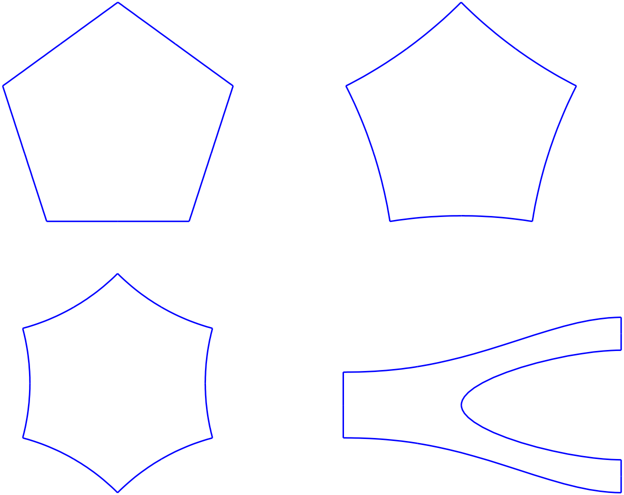
$$\mathbf{x} = \mathbf{x}_\xi + \mathbf{x}_\eta - \mathbf{x}_{\xi\eta}$$

Transfinite extension (a.k.a. the Gordon-Hall algorithm)

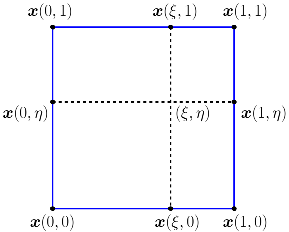




Generalized transfinite extension



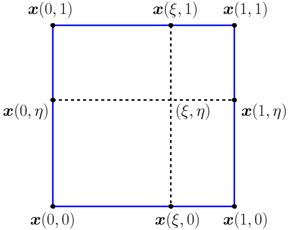
Generalized transfinite extension



Transfinite extension:

$$\begin{aligned}
\mathbf{x}(\xi,\eta) &= \phi_1(\xi,\eta)\mathbf{x}(0,\eta) + \phi_3(\xi,\eta)\mathbf{x}(1,\eta) \\
&+ \phi_2(\xi,\eta)\mathbf{x}(\xi,0) + \phi_4(\xi,\eta)\mathbf{x}(\xi,1) \\
&- \sum_{i=1}^{4} \phi_i(\xi,\eta)\phi_{i+1}(\xi,\eta)\mathbf{x}_i
\end{aligned}$$

Generalized transfinite extension



Transfinite extension:

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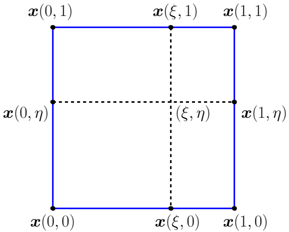
\end{aligned}
$$

$$
\mathbf{x}(\xi,\eta) = \sum_{i=1}^{4} \phi_i(\xi,\eta)\mathbf{x}(\pi_i(\xi,\eta)) - \phi_i(\xi,\eta)\phi_{i+1}(\xi,\eta)\mathbf{x}_i
$$

Generalized transfinite extension



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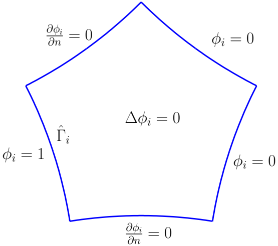
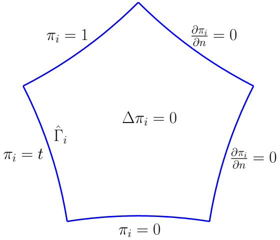
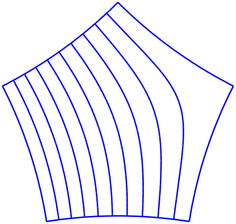
\end{aligned}
$$

$$
\mathbf{x}(\xi,\eta) = \sum_{i=1}^{n} \phi_i(\xi,\eta)\mathbf{x}(\pi_i(\xi,\eta)) - \phi_i(\xi,\eta)\phi_{i+1}(\xi,\eta)\mathbf{x}_i
$$

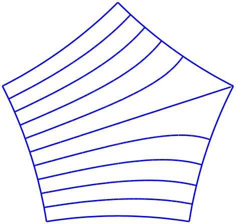
Generalized transfinite extension







Weight function: ϕ_i



Projection function: π_i

Regularity

Spectral element approximation

Exponential convergence when u is analytic

$$\|u - u_{\mathcal{N}}\|_{H^1} \leq e^{-\mathcal{N}}$$

Algebraic convergence when u is less regular, e.g., $u \in H^\sigma$

$$\|u - u_{\mathcal{N}}\|_{H^1} \leq c\mathcal{N}^{1-\sigma}\|u\|_{H^\sigma}$$

Regularity

Dirichlet/neumann problem on cornered domain

For the Laplace problem, $\Delta u = 0$, with mixed Dirichlet/Neumann boundary conditions

$$u \in H^{1+\frac{\pi}{2\omega}-\epsilon}$$

where $\omega \neq \frac{\pi}{2}$ is the largest angle in the domain, and ϵ is a small positive constant. This gives the convergence estimate

$$\|u - u_{\mathcal{N}}\|_{H^1} \leq c\mathcal{N}^{\epsilon-2\frac{\pi}{2\omega}}$$

Regularity

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$$\|u - u_{\mathcal{N}}\|_{H^1} \leq c\mathcal{N}^{\epsilon-2\frac{\pi}{2\omega}}$$

Dirichlet problem on cornered domain

For the Laplace problem with only Dirichlet boundary conditions

$$u \in H^{1+\frac{\pi}{\omega}-\epsilon}$$

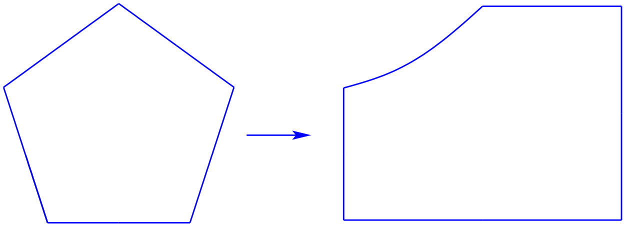
and corresponding convergence estimate

$$\|u - u_{\mathcal{N}}\|_{H^1} \leq c\mathcal{N}^{\epsilon-2\frac{\pi}{\omega}}$$

Regularity

Reference pentagon with $\omega = \frac{3\pi}{5}$ gives

$$\|u - u_{\mathcal{N}}\|_{H^1} \leq c\mathcal{N}^{\epsilon - \frac{5}{3}} \qquad \text{and} \qquad \|u - u_{\mathcal{N}}\|_{H^1} \leq c\mathcal{N}^{\epsilon - \frac{10}{3}}$$

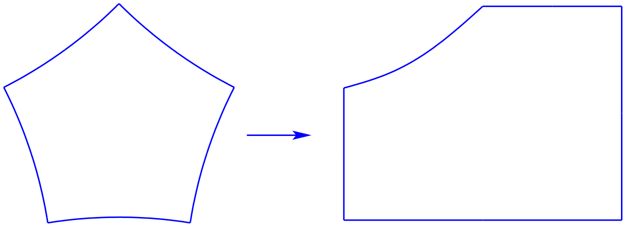


Method	Convergence rate
Transfinite extension	3.7
Harmonic extension	4.3
Pseudo-harmonic extension	2.5
Mean value extension	2.5

Results

Reference pentagon with $\omega = \frac{\pi}{2}$ gives

$$\|u - u_{\mathcal{N}}\|_{H^1} \leq c\mathcal{N}^{\epsilon - 4} \quad \text{and} \quad \|u - u_{\mathcal{N}}\|_{H^1} \leq c\mathcal{N}^{\epsilon - 4}$$

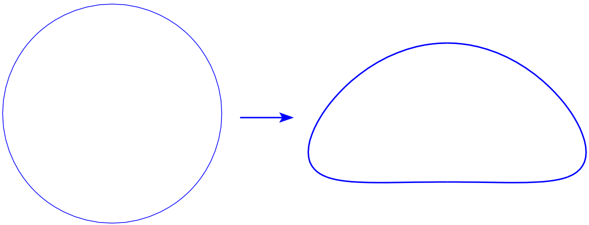


Method	Convergence rate
Transfinite extension	5.0
Harmonic extension	5.0

Results

Reference circle mapped to deformed ellipse defined by

$$(x, y) = \Phi(\xi, \eta) = (a\xi, b\eta - \delta(\xi^2 - \eta^2))$$

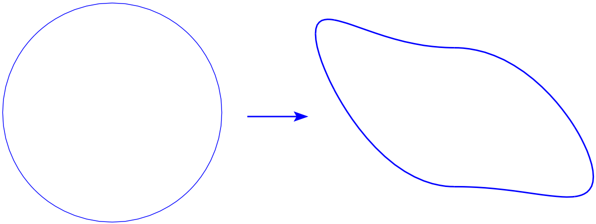


Method	Convergence rate
Harmonic extension	∞
Pseudo-harmonic extension	∞
Mean-value interpolation	4.6

Results

Reference circle mapped to domain with C^1-boundary defined by

$$
\begin{aligned}
(x, y) &= (a\xi, b\eta - \delta(\xi^2 - \eta^2)) && \xi \geq 0 \\
(x, y) &= (a\xi, b\eta - \delta(\xi^2 - \eta^2) + (a - 0.2)^2\xi^2) && \xi < 0
\end{aligned}
$$



Method	Convergence rate
Harmonic extension	5
Pseudo-harmonic extension	5
Mean-value interpolation	4.6

Conclusions

- For large deformations the harmonic extension produces folding grids.
- On cornered domains, the generalized transfinite extension is a good alternative.
- On smooth domains, the barycentric extensions are good alternatives.
- Both barycentric extension and transfinite extension is faster than the harmonic extension.
- In $3D$ a combination of different methods can be used.