Stabilized reduced basis approximation of the Navier-Stokes equations in deformed domains

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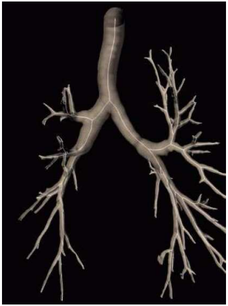
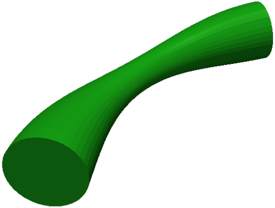
Motivation





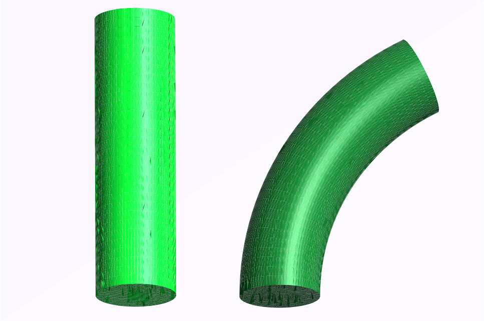
Outline

- Motivation
- Geometry as a parameter in Navier-Stokes
- Stabilized $P_1 - P_1$ finite elements
- Reduced basis procedure
- LifeV implementation
- Results

Parametrization

For some parameter $\boldsymbol{\mu} \in \mathbb{R}^P$, we define a map

$$\Phi(\boldsymbol{\mu}) : \widehat{\Omega} \to \Omega_{\boldsymbol{\mu}}.$$



Parametrization

On $\Omega_{\boldsymbol{\mu}}$ we define the operators

$$a(\mathbf{v}, \mathbf{w}; \boldsymbol{\mu}) = \nu \int_{\Omega_{\boldsymbol{\mu}}} \nabla \mathbf{v} \cdot \nabla \mathbf{w} d\Omega_{\boldsymbol{\mu}},$$

$$b(\mathbf{v}, q; \boldsymbol{\mu}) = -\int_{\Omega_{\boldsymbol{\mu}}} q \nabla \cdot \mathbf{v} d\Omega_{\boldsymbol{\mu}},$$

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}; \boldsymbol{\mu}) = \int_{\Omega_{\boldsymbol{\mu}}} (\mathbf{u} \cdot \nabla)\mathbf{v}\, \mathbf{w} d\Omega_{\boldsymbol{\mu}},$$

where $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are velocity fields, q is a scalar pressure field, and ν is the viscosity.

Parametrization

To identify the geometry as a parameter we write the viscous operator on the reference domain:

$$a(\mathbf{u}, \mathbf{v}; \boldsymbol{\mu}) = \nu \int_{\widehat{\Omega}} \mathcal{J}^{-T} \widehat{\nabla}(\mathbf{u} \circ \Phi) \cdot \mathcal{J}^{-T} \widehat{\nabla}(\mathbf{v} \circ \Phi) |J| d\widehat{\Omega},$$

where $\mathcal{J}$ is the Jacobian of the map.

Parametrization

The weak form of the Navier-Stokes equations on $\Omega_{\boldsymbol{\mu}}$ then reads: find $(\mathbf{u}, p) \in V(\Omega_{\boldsymbol{\mu}})$ such that

$$\left.\begin{array}{rcl} a(\mathbf{u}, \mathbf{v}; \boldsymbol{\mu}) + b(\mathbf{v}, p; \boldsymbol{\mu}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}; \boldsymbol{\mu}) & = & l(\mathbf{v}; \boldsymbol{\mu}) \\ b(\mathbf{u}, q; \boldsymbol{\mu}) & = & 0 \end{array}\right\} \forall (\mathbf{v}, q) \in V(\Omega_{\boldsymbol{\mu}}),$$

where $V(\Omega_{\boldsymbol{\mu}}) = \{\mathbf{v} \in (H^1(\Omega_{\boldsymbol{\mu}}))^3 : \mathbf{v}|_{\Gamma^D} = 0\} \times \{q \in L^2(\Omega_{\boldsymbol{\mu}})\} = X \times M$.

The inf-sup condition for the Navier-Stokes equations

$$0 < \beta \leq \inf_{q \in M} \sup_{\mathbf{v} \in X} \frac{b(\mathbf{v}, q; \boldsymbol{\mu})}{\|q\|_{L^2} \|\mathbf{v}\|_{H^1}}$$

Stabilized $P_1 - P_1$ finite elements

We define the space

$$V_h(\Omega_{\boldsymbol{\mu}}) = \{(\mathbf{v}_h, p_h) \in V(\Omega_{\boldsymbol{\mu}}) : \mathbf{v}|_K \in P_1 \text{ and } p|_K \in P_1 \ \forall K \in \mathcal{T}_h\},$$

where P_1 represents first order polynomials,
and the stabilization term

$$j_h(p, q; \boldsymbol{\mu}, \mathbf{z}) = \int_{\Gamma_{I,\boldsymbol{\mu}}} \gamma \frac{h_f^3}{\max\{h_f|\mathbf{z}|, \nu\}} [\![\nabla p]\!]_f \cdot [\![\nabla q]\!]_f \, ds.$$

[Burman, Fernández, Hansbo , 2006.]

Stabilized $P_1 - P_1$ finite elements

Finite element problem: find $(\mathbf{u}_h, p_h)$ in $V_h(\Omega_{\boldsymbol{\mu}})$ such that

$$\mathcal{A}_h[(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h), \mathbf{u}_h; \boldsymbol{\mu}] = l_h(\mathbf{v}_h; \boldsymbol{\mu}) \quad \forall (\mathbf{v}_h, q_h) \in V_h(\Omega_{\boldsymbol{\mu}}),$$

where subscript h on the operators means integration by some appropriate quadrature rule, and

$$\begin{aligned}\mathcal{A}_h[(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h), \mathbf{u}_h; \boldsymbol{\mu}] = {} & a_h(\mathbf{u}_h, \mathbf{v}_h; \boldsymbol{\mu}) + b_h(\mathbf{v}_h, p_h; \boldsymbol{\mu}) - b_h(\mathbf{u}_h, q_h; \boldsymbol{\mu}) \\ & + c_h(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h; \boldsymbol{\mu}) + j_h(p_h, q_h; \boldsymbol{\mu}, \mathbf{u}_h).\end{aligned}$$

Stabilized $P_1 - P_1$ finite elements

The non-linear system is solved by a pseudo time advancing scheme until a steady solution is reached. We introduce a time step Δt and the bi-linear form

$$m(\mathbf{u}, \mathbf{v}; \boldsymbol{\mu}) = \frac{1}{\Delta t} \int_{\Omega_{\boldsymbol{\mu}}} \mathbf{u} \cdot \mathbf{v} d\Omega_{\boldsymbol{\mu}}.$$

Starting from an initial guess $(\mathbf{u}_h^0, p_h^0)$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1})$ in $V_h(\Omega_{\boldsymbol{\mu}})$ such that

$$m_h(\mathbf{u}_h^{n+1}, \mathbf{v}_h; \boldsymbol{\mu}) + \mathcal{A}_h[(\mathbf{u}_h^{n+1}, p_h^{n+1}), (\mathbf{v}_h, q_h), \mathbf{u}_h^n; \boldsymbol{\mu}] = l_h(\mathbf{v}_h; \boldsymbol{\mu}) + m_h(\mathbf{u}_h^n, \mathbf{v}_h; \boldsymbol{\mu}) \qquad \forall (\mathbf{v}_h, q_h) \in V_h(\Omega_{\boldsymbol{\mu}}).$$

Reduced basis: offline procedure

Compute the basis functions

For a parameter set $S_N = \{\boldsymbol{\mu}^i\}_{i=1}^N$, each $\boldsymbol{\mu}^i$ defines a map $\Phi^i : \widehat{\Omega} \to \Omega_{\boldsymbol{\mu}^i}$. On each $\Omega_{\boldsymbol{\mu}^i}$ we find $(\mathbf{u}_h^i, p_h^i) \in V_h(\Omega_{\boldsymbol{\mu}^i})$ such that

$$\mathcal{A}_h[(\mathbf{u}_h^i, p_h^i), (\mathbf{v}_h, q_h), \mathbf{u}_h^i; \boldsymbol{\mu}^i] = l_h(\mathbf{v}_h; \boldsymbol{\mu}^i) \quad \forall (\mathbf{v}_h, q_h) \in V_h(\Omega_{\boldsymbol{\mu}^i}).$$

Map the basis functions to $\widehat{\Omega}$

The Piola transformation of $\mathbf{u}^i$ from $\Omega_{\boldsymbol{\mu}^i}$ to $\widehat{\Omega}$

$$\hat{\mathbf{u}}^i = \Psi^i(\mathbf{u}^i) = (\mathcal{J}^i)^{-1}(\mathbf{u}^i \circ \Phi^i)|J^i|.$$

The pressure is mapped through $\hat{p}^i = p^i \circ \Phi^i$.

Orthonormalize

Pressure basis wrt the L^2 inner product, velocity wrt $a_h(\mathbf{v}, \mathbf{w}; \boldsymbol{\mu}^0)$

Reduced basis: online procedure

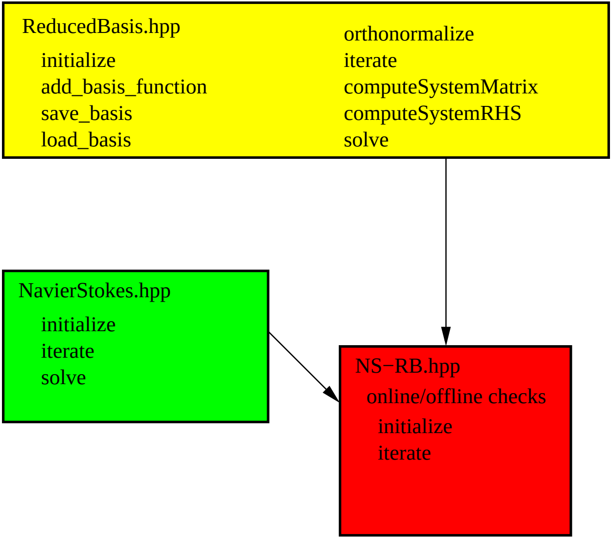
Map the basis functions to Ω

$$\tilde{\mathbf{u}}^i = \Psi^{-1}(\hat{\mathbf{u}}^i) \text{ and } \tilde{p}^i = \hat{p}^i \circ \Phi^{-1}(\boldsymbol{\mu}), \quad i = 1, \dots, N.$$

Start with $(\mathbf{u}_N^0, p_N^0) = (\mathbf{0}, \mathbf{0})$ and repeat for each time-step

- Compute the system matrix for $(\mathbf{u}_N^n, p_N^n)$
- Solve the resulting linear system for α_i^{n+1} and β_i^{n+1}
- Assemble $\mathbf{u}_N^{n+1} = \sum_{i=1}^N \alpha_i^{n+1} \tilde{\mathbf{u}}^i$ and $p_N^{n+1} = \sum_{i=1}^N \beta_i^{n+1} \tilde{p}^i$.
- Stop when $\|(\mathbf{u}_N^{n+1}, p_N^{n+1}) - (\mathbf{u}_N^n, p_N^n)\|_{H^1 \times L^2} < \epsilon$.

Implementation in LifeV



Implementation in LifeV

On a straight reference pipe, we impose deformation controlled by two parameters μ_1 and μ_2. First

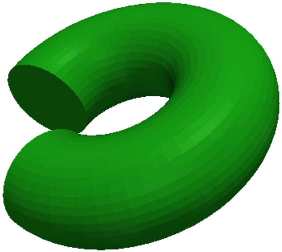
$$r(l) = r_0 + \frac{\mu_1}{2}(\cos(\frac{2\pi l}{L}) - 1)$$

defines the change in cross-section radius along the length of the pipe.

Implementation in LifeV

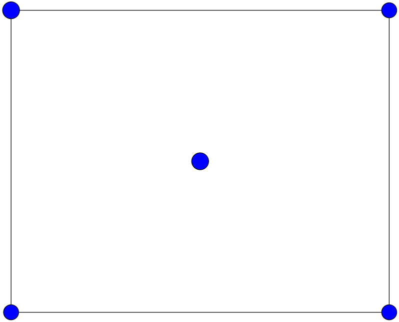
Assuming the center axis is aligned with the z-axis, the center axis is rotated around the point $(\frac{1}{\mu_2}, 0, 0)$. For a point (x_0, y_0, z_0) the second deformation gives

$$
\begin{aligned}
x &= \frac{1}{\mu_2}(1 - \cos(\theta)) + x_0 \cos(\theta) \\
y &= y_0 \\
z &= \frac{1}{\mu_2}\sin(\theta) - x_0 \sin(\theta),
\end{aligned}
$$



Results

We let $\mu_1 \in [0, 0.4]$ and $\mu_2 \in [0, 1.0]$ and choose the parameter set S_N.



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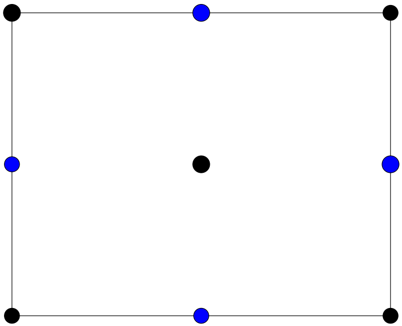

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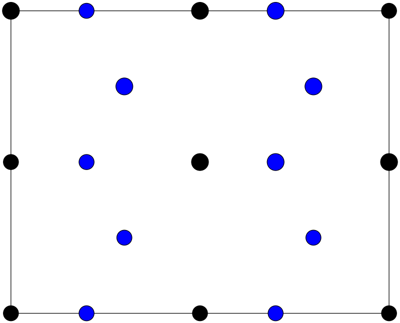

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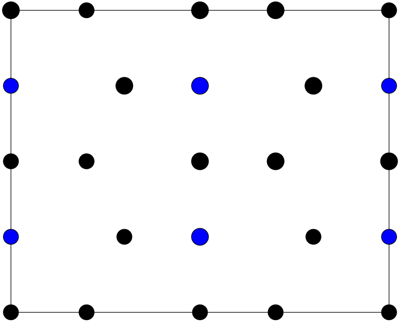

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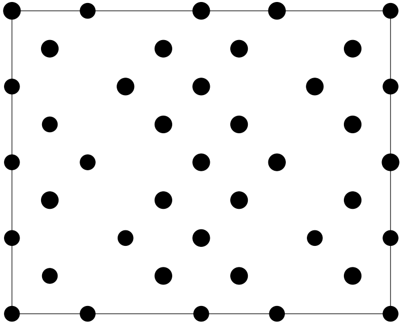
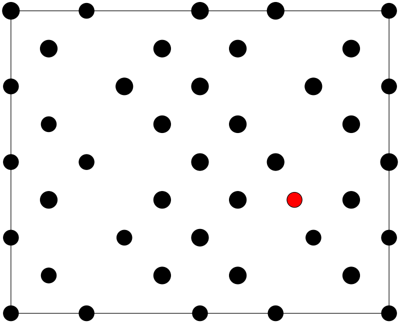


Results

As a test case we choose $\boldsymbol{\mu} = (0.15, 0.75)$



Results

N	$\|p - p_N\|_{L^2}$	$\|\mathbf{u} - \mathbf{u}_N\|_{H^1}$
5	$1.78 \cdot 10^{-1}$	$2.53 \cdot 10^{-4}$
10	$2.55 \cdot 10^{-2}$	$1.69 \cdot 10^{-4}$
15	$1.51 \cdot 10^{-2}$	$1.21 \cdot 10^{-4}$
20	$5.31 \cdot 10^{-3}$	$7.44 \cdot 10^{-5}$
25	$4.83 \cdot 10^{-3}$	$6.99 \cdot 10^{-5}$
30	$4.46 \cdot 10^{-3}$	$6.66 \cdot 10^{-5}$
35	$3.57 \cdot 10^{-3}$	$5.95 \cdot 10^{-5}$
40	$3.12 \cdot 10^{-3}$	$5.61 \cdot 10^{-5}$

The convergence of the reduced basis approximation with respect to the number of basis functions.

Final comments - future work

- Stabilization works
- No enrichment necessary
- Offline/online decomposition
- Error estimation
- Greedy algorithm