Piecewise Kriging in Fields with Varying Anisotropy

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PIECEWISE KRIGING IN FIELDS WITH VARYING ANISOTROPY - WORKING PAPER

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Abstract

Several proposals have been made to handle kriging in fields where the anisotropy is not constant but varies from location to location in the field in question. This note proposes an alternative approach for dealing with this problem. The basic idea is to apply so-called piecewise kriging along shortest paths between the locations where the function to be estimated are known and the location where the function is to be estimated. The proposed approach avoids the problem of establishing positive definite covariance matrices in the whole field.

1 Introduction

Several proposals have been made to handle kriging in fields where the anisotropy is not constant but varies from location to location in the field in question. See for example [1] and the references cited there. This note proposes an alternative approach for dealing with this problem. The basic idea is to apply so-called piecewise kriging along shortest paths between the locations where the function to be estimated are known and the location where the function is to be estimated. Our approach avoids the problem of establishing positive definite covariance matrices in the whole field.

2 Piecewise simple kriging under the assumption of isotropy

This section is included in order to motivate the proposed procedure for the anisotropic case. The following notation is used:

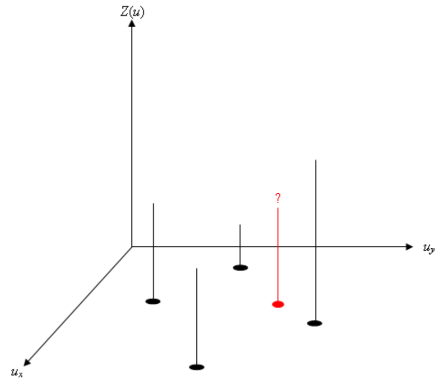

Figure 1: Observed values and sought value (two dimensions)

u:	Vector of spatial coordinates in a finite-dimensional space
$Z(u)$:	Random field value to be sought
$\widehat{Z}(u)$:	Estimate of $Z(u)$
$z(u_i)$:	Observation of $Z(u)$ at $z(u_i)$
h :	Vector between two spatial locations

The situation is illustrated in Figure 1 in the two-dimensional case. We shall make the standard simple kriging assumptions which imply that the expected value and variance of $Z(u)$ are independent of u and known. Thus we can normalize the observations and assume that $Z(u)$ has expected value 0 and variance 1.
The covariance (and correlation coefficient) $\rho(h)$ between $Z(u)$ and $Z(u+h)$ is given as:

$$\rho(h) = \mathrm{E}[Z(u)(Z(u+h)] \tag{1}$$

If we assume isotropy, $\rho(h)$ depends on h through the Euclidean metric $|h|$ only, so that we can operate with $\rho(|h|)$.

The variogram at the point u is:

$$\gamma(|h|) = \frac{1}{2}\mathrm{E}[(Z(u+h) - Z(u)]^2 = 1 - \rho(|h|). \tag{2}$$

Assume that we have only one observation $z(u_1)$. Then the simple kriging estimate of $Z(u)$ becomes

$$\widehat{Z}(u) = \rho(|u_1 - u|)z(u_1). \tag{3}$$

Now we put an intermediate point v_{11} somewhere on the chord between u_1 and u. Then, as above we get

$$\widehat{Z}(v_{11}) = \rho(|u_1 - v_{11}|)z(u_1). \tag{4}$$

Let us for a moment assume that $\widehat{Z}(v_{11})$ is actually the true value of $Z(v_{11})$, and that we want to estimate the the value of $Z(u)$ based solely on $\widehat{Z}(v_{11})$. We would then get

$$\begin{aligned}\widehat{Z}(u) = \rho(|v_{11} - u|)\widehat{Z}(v_{11}) = \\ \rho(|v_{11} - u|)\rho(|u_1 - v_{11}|)z(u_1).\end{aligned} \tag{5}$$

We see that $\widehat{Z}(u)$ is still (unconditionally) unbiased. If $\rho(|h_1| + |h_2|) = \rho(|h_1|)\rho(|h_2|)$ we would be back to simple kriging. This condition holds, however, only if the corresponding variogram is exponential, i.e. if $\rho(|h|)$ is of the form $\exp(-a|h|/r)$. So, in general, the kriging variance will be greater than in simple kriging.
We can continue this reasoning by placing a new intermediate point v_{12} between v_{11} and u. As above we assume that $\widehat{Z}(v_{11})$ is the true value of $Z(v_{11})$. First we estimate $Z(v_{12})$ based solely on $\widehat{Z}(v_{11})$. As in (5) we get

$$\widehat{Z}(v_{12}) = \rho(|v_{11} - v_{12}|)\widehat{Z}(v_{11}). \tag{6}$$

Again we assume that $\widehat{Z}(v_{12})$ is the true value of $Z(v_{12})$, and that we want to estimate $Z(u)$ based solely on $\widehat{Z}(v_{12})$. We then get

$$\begin{aligned}\widehat{Z}(u) = \rho(|v_{12} - u|)\widehat{Z}(v_{12}) = \\ \rho(|v_{12} - u|)\rho(|v_{11} - v_{12}|)\widehat{Z}(v_{11}) = \\ \rho(|v_{12} - u|)\rho(|v_{11} - v_{12}|)\rho(|u_1 - v_{11}|)z(u_1).\end{aligned} \tag{7}$$

So again, if $\rho(|h|) = \exp(-a|h|/r)$ we would be back to simple kriging.

In general, if we place m_1 intermediate points $v_{11}, \cdots, v_{1m_1}$ on the chord between u_1 and u, the form of the modified estimate $\widehat{Z}(u)$ becomes

$$\widehat{Z}(u) = \rho(|u_1 - v_{11}|)[\prod_{j=1}^{m_1-1} \rho(|v_{1j} - v_{1,j+1}|)]\rho(|v_{1m_1} - u|)z(u_1). \tag{8}$$

Now the question arises, how much does this differ from the simple kriging estimate when the distance between u_1 and u is small compared to minimum of the ranges of the variograms that we used on the pieces of the path between u_1 and u? In other words, how much does $\rho(\sum_{j=1}^{m_1} |h_j|)$ differ from $\prod_{j=1}^{m_1} \rho(|h_j|)$ when the $|h_j|$-s are small? It is well known (see e.g.[2] p. 62) that for valid variograms

$$\rho(|h|) = 1 - a|h|/r + \mathrm{o}(|h|^2/r^2) \tag{9}$$

where r is the range of the corresponding variogram. Then, if we set $h = \sum_{j=1}^{m_1} |h_j|$,

$$\rho(|h|) = 1 - a|h|/r + \mathrm{o}(|h|^2/r^2) \tag{10}$$

and

$$\prod_{j=1}^{m_1} \rho(|h_j|) = \prod_{j=1}^{m_1} [1 - a|h_j|/r + \mathrm{o}(|h_j|/r)] =$$
$$1 - a\sum_{j=1}^{m_1} |h_j|/r + \mathrm{o}[(\max_{j=1,\cdots,m_1} |h_j|)^2/r^2] \tag{11}$$

so that

$$|\rho(\sum_{j=1}^{m_1} |h_j|) - \prod_{j=1}^{m_1} \rho(|h_j|)| = \mathrm{o}(h^2/r^2). \tag{12}$$

This means that if $|u_1 - u|$ is small compared to the range r, our modification does not deviate too much from simple kriging.

3 Piecewise simple kriging, varying anisotropy

We now assume that the m_1 intermediate points $v_{11}, \cdots, v_{1m_1}$ do not necessarily lie on the chord between u_1 and u, and that we have established m_1 relevant, in general not equal, variograms, together with corresponding, in general non-Euclidean, metrics for the estimation of $Z(v_{11})$ from $z(u_1)$ and for the estimation of $Z(v_{1,j+1})$ from $Z(v_{1j})$ for $j = 1, \cdots, m_1 - 1$. We still propose to use the procedure described in section 2 on page 1 for estimating $Z(v_{1m_1})$.

Now we generalize to the case where we possess observations $z(u_i)$ for a set of points $u_1, \cdots, u_n$, and where we for each i have found a path $(u_i, v_{i1}, \cdots, v_{im_i}, u)$ between u_i and u with valid local variograms for each of the chords $(u_i, v_{i1}), (v_{i1}, v_{i2}), \cdots, (v_{i,m_{i-1}}, v_{im_i}), (v_{im_i}, u)$. We assume further that we have chosen the points v_{im_i} so close to u that we can use the same variogram for all the chords $(v_{im_i}, u), i = 1, \cdots, n$. Based on the estimates $\widehat{Z}(v_{im_i}), i = 1, \cdots, n$ we then use simple or ordinary kriging to estimate $Z(u)$.

4 Selection of the path from u_i to u

We want the path from u_i to u to be short. First we partition the u-space into rectangular cells such that we can establish an in general anisotropic, multidimensional variogram for each cell. Then we put nodes on the edges of the cells as illustrated for two dimensions in Figure 2 on the following page. The extension to higher dimensions is obvious.

Every node on an edge of one cell is connected with a straight line to every other node on an edge of the same cell. In addition, the starting point u_i

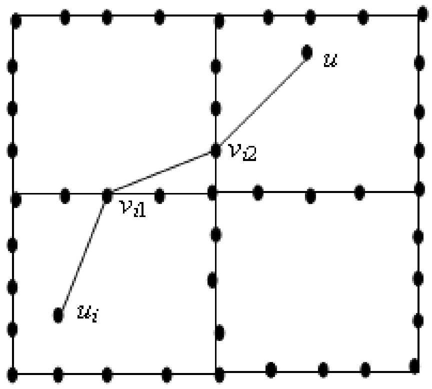

Figure 2: Illustration of the shortest path calculation in the two-dimensional case

and the ending point u are connected to every edge node in their own cells by straight lines. A cell's local metric is established in the usual way by a local change of coordinates. This metric is used to calculate the lengths of the lines in the cells. Then Dijksra's algorithm is used to establish a shortest path from u to all the u_i-s in one pass. The number of nodes along the edges is determined by the accuracy required. Of course, the number of lines in the network increases rapidly with the dimension of the u-space.

5 Extension to the case where $\mathrm{E}Z(u)$ is unknown but still independent of u

This is an assumption used in ordinary kriging. Using the ordinary kriging equations to estimate $Z(v_{i1})$ based on $z(u_i)$ gives only the trivial estimate $\widehat{Z}(v_{i1}) = Z(u_i)$ which is not very helpful. We propose instead to continue with simple kriging by normalizing $Z(u)$ around the empirical mean $\sum_{i=1}^{n} z(u_i)/n$ in lieu of around a known $\mathrm{E}Z(u)$, and then use simple kriging as above to establish $\widehat{Z}(v_{im_i})$
Once we have established $\widehat{Z}(v_{im_i}), i = 1, \cdots, n$ we use simple or ordinary kriging to estimate $Z(u)$ based on these values.

6 A two-dimensional illustrative example

We have four observations $z(u_1) = 3$, $z(u_2) = -1$, $z(u_3) = -2$, $z(u_4) = 5$ of the variable Z defined on the field shown in Figure 3 on the next page. We assume that the observations have been normalized so that $\mathrm{E}Z = \mu = 0$ and $\mathrm{var}Z = 1$, and that the anisotropy is constant in each of the cells (1,1), $\cdots$,

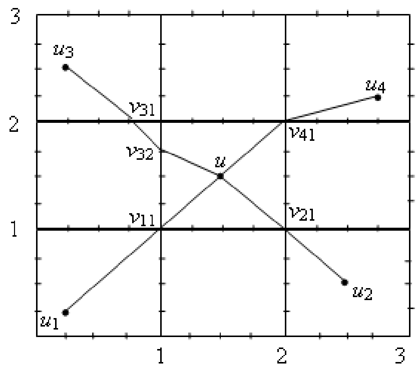

Figure 3: The illustrative example

(3,3). The derivation of the distance function from the local anisotropies in the cells is omitted, see e.g. [1]. The distance functions have been calculated to be:

$$\begin{aligned}
d_{11}(h_1,h_2) &= \sqrt{h_1^2 - 2h_1h_2 + 4h_2^2} \\
d_{21}(h_1,h_2) &= \sqrt{h_1^2 + 4h_2^2} \\
d_{31}(h_1,h_2) &= \sqrt{h_1^2 + 2h_1h_2 + 4h_2^2} \\
d_{12}(h_1,h_2) &= \sqrt{h_1^2 - 2h_1h_2 + 3h_2^2} \\
d_{22}(h_1,h_2) &= \sqrt{h_1^2 + 3h_2^2} \\
d_{32}(h_1,h_2) &= \sqrt{h_1^2 + 2h_1h_2 + 3h_2^2} \\
d_{13}(h_1,h_2) &= \sqrt{h_1^2 - 2h_1h_2 + 2h_2^2} \\
d_{23}(h_1,h_2) &= \sqrt{h_1^2 + 2h_2^2} \\
d_{33}(h_1,h_2) &= \sqrt{h_1^2 + 2h_1h_2 + 2h_2^2}.
\end{aligned} \tag{13}$$

The shortest paths are shown in Figure 3. Their lengths are:

Between u and u_1: 2.2990
Between u and u_2: 2.3229
Between u and u_3: 1.5150
Between u and u_4: 2.0308.

We choose for all cells the Gaussian variogram with generic form:

$$\gamma(|h|) = s[1 - \exp(-3|h|^2/r^2)^3] \text{ with } r = 3 \text{ and } s = 1. \tag{14}$$

This gives

$$\rho(|h|) = \exp(-|h|^2/3)^3 \tag{15}$$

where $|h|$ is to be replaced by the values of the distance function in each individual cell. This gives

$$\begin{aligned}
\widehat{Z}(v_{11}) &= \rho[d(u_1, v_{11})]z(u_1) = 2.511 \\
\widehat{Z}(v_{21}) &= \rho[d(u_2, v_{21})]z(u_2) = -0.820 \\
\widehat{Z}(v_{31}) &= \rho[d(u_3, v_{31})]z(u_3) = -2.000 \\
\widehat{Z}(v_{32}) &= \rho[d(v_{31}, v_{32})]z(v_{31}) = -2.000 \\
\widehat{Z}(v_{41}) &= \rho[d(u_4, v_{41})]z(u_4) = 4.783.
\end{aligned} \tag{16}$$

Finally, to obtain $\widehat{Z}(u)$ we use simple kriging in cell (2,2) based on $\widehat{Z}(v_{11})$, $\widehat{Z}(v_{21})$, $\widehat{Z}(v_{32})$, and $\widehat{Z}(v_{41})$:

$$\widehat{Z}(u) = \lambda_1\widehat{Z}(v_{11}) + \lambda_2\widehat{Z}(v_{21}) + \lambda_3\widehat{Z}(v_{32}) + \lambda_4\widehat{Z}(v_{41}) \tag{17}$$

where the λ-s satisfy

$$\begin{aligned}
\lambda_1 + \rho(v_{11}, v_{21})\lambda_2 + \rho(v_{11}, v_{32})\lambda_3 + \rho(v_{11}, v_{41})\lambda_4 &= \rho(v_{11}, u) \\
\rho(v_{21}, v_{11})\lambda_1 + \lambda_2 + \rho(v_{21}, v_{32})\lambda_3 + \rho(v_{21}, v_{41})\lambda_4 &= \rho(v_{21}, u) \\
\rho(v_{32}, v_{11})\lambda_1 + \rho(v_{32}, v_{21})\lambda_2 + \lambda_3 + \rho(v_{32}, v_{41})\lambda_4 &= \rho(v_{32}, u) \\
\rho(v_{41}, v_{11})\lambda_1 + \rho(v_{41}, v_{21})\lambda_2 + \rho(v_{41}, v_{32})\lambda_3 + \lambda_4 &= \rho(v_{41}, u).
\end{aligned} \tag{18}$$

This gives $\lambda_1 = -0.071$, $\lambda_2 = 0.0721$, $\lambda_3 = 0.3659$, $\lambda_4 = 0.3610$, and $\widehat{Z}(u) = 0.2238$.

References

[1] J. B. Boisvert, J. G. Manchuk, and C. V. Deutsch. Kriging in the Presence of Locally Varying Anisotropy Using Non-Euclidean Distances. *Math Geol*, 41(5):585–601, 2009.

[2] N. A. C. Cressie. *Statistics for Spacial Data.* New York: John Wiley, 1993.