The Minimal Residual Method applied to ill-posed optimality systems; eigenvalues and convergence properties

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Inverse problem

$$\min_{v,u} \left\{ \frac{1}{2} \left\| Tu - d \right\|^2 + \frac{1}{2} \alpha \left\| v \right\|^2 \right\}$$

subject to

$$Au = -Bv$$

- Bounded linear operators:

$$B : H_1 \to H_2$$
$$A : H_2 \to H_2$$
$$T : H_2 \to H_3$$

Optimality system

$$\begin{bmatrix} \alpha I & 0 & B^* \\ 0 & T^*T & A^* \\ B & A & 0 \end{bmatrix} \begin{bmatrix} v \\ u \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ T^*d \\ 0 \end{bmatrix}$$

- ill posed $\alpha = 0$
- ill conditioned $0 < \alpha \ll 1$
- Minimal Residual Method

Main result

- $$\mathcal{A}_\alpha = \begin{bmatrix} \alpha I & 0 & B^* \\ 0 & T^*T & A^* \\ B & A & 0 \end{bmatrix}$$

- $\mathrm{sp}(\mathcal{A}_\alpha) \subset [-b, -a] \cup [c\alpha, d\alpha] \cup \{\lambda_1, \lambda_2, \ldots, \lambda_{N(\alpha)}\} \cup [a, b]$

- $N(\alpha) = O(\ln(\alpha^{-1}))$

- MINRES iterations $O(\ln(\alpha^{-1}))$

Example

$$\min_{v,\,u} \left\{ \frac{1}{2} \left\| u - d \right\|^2_{L^2(\partial\Omega)} + \frac{1}{2}\alpha \left\| v \right\|^2_{L^2(\Omega)} \right\}$$

subject to

$$\begin{aligned} -\nabla \cdot (k \nabla u) + u &= \begin{cases} -v & \text{in } D = (0.25, 0.75) \times (0.25, 0.75) \\ 0 & \text{in } \Omega \setminus D \end{cases} \\ \nabla u \cdot \vec{n} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

where

$$k(x, y) = 2 + \sin(2\pi(x + y))$$

Example, continued

- Not covered by our framework because

$$A : H^1(\Omega) \to (H^1(\Omega))'$$

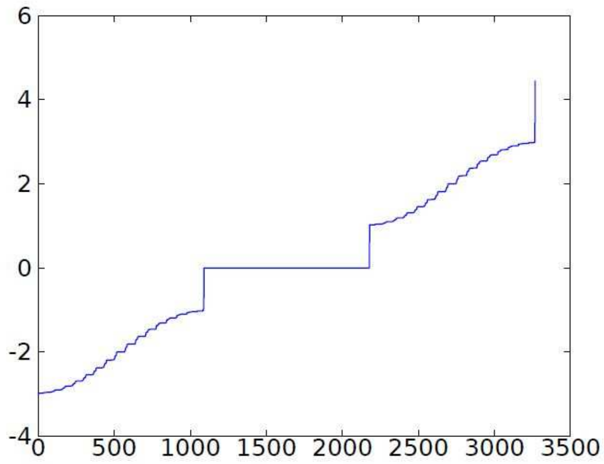
- Riesz map R,

$$R^{-1}Au = -R^{-1}Bv$$

- MG approximation of R^{-1}
- Note

$$R^{-1}A : H^1(\Omega) \to H^1(\Omega)$$

Example, continued



Eigenvalues of $\begin{bmatrix} \alpha I & 0 & B^* \\ 0 & T^*T & A^* \\ B & A & 0 \end{bmatrix}$ with $\alpha = 0.001$

Example, continued

$N\backslash\alpha$	1.0	0.1	0.01	0.001	0.0001
8	71	100	49	55	54
16	91	99	72	49	61
32	139	125	108	91	91
64	134	129	115	85	81
128	130	143	78	91	94
256	160	155	148	122	111
512	109	122	97	128	123

Table 1: MINRES iterations

Analysis

- $\mathcal{A}_\alpha = \begin{bmatrix} \alpha I & 0 & B^* \\ 0 & T^*T & A^* \\ B & A & 0 \end{bmatrix}$
- Negative eigenvalues:

$$[-b, -a],$$

$b, a > 0$ independent of α

Analysis, continued

- Positive eigenvalues

$$
\begin{aligned}
\mathcal{A}_\alpha &= \begin{bmatrix} 0 & 0 & B^* \\ 0 & T^*T & A^* \\ B & A & 0 \end{bmatrix} + \begin{bmatrix} \alpha I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \mathcal{A}_0 + \mathcal{E}_\alpha,
\end{aligned}
$$

  - Severely ill posed: $|\lambda_i(\mathcal{A}_0)| \leq c\, e^{-Ci}$
  - Courant-Fischer-Weyl min-max principle
- Chebyshev polynomials

Summary

$$\begin{bmatrix} \alpha I & 0 & B^* \\ 0 & T^*T & A^* \\ B & A & 0 \end{bmatrix} \begin{bmatrix} v \\ u \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ T^*d \\ 0 \end{bmatrix}$$

- MINRES good alternative
- Spectrum

$$[-b, -a] \cup [c\alpha, d\alpha] \cup \{\lambda_1, \lambda_2, \ldots, \lambda_{N(\alpha)}\} \cup [a, b]$$

- Iterations $O(\ln(\alpha^{-1}))$

Comparison

- Schöberl and Zulehner
- Random initial guess, true solution $(v, u, w) = 0$
- Stop: $\|(v, u, w)_k\| \leq 10^{-4}$

$$\min_{v,u} \left\{ \frac{1}{2} \|u - d\|^2_{L^2(\Omega)} + \frac{1}{2} \alpha \|v\|^2_{L^2(\Omega)} \right\}$$

subject to

$$\begin{aligned} -\Delta u + u &= v \text{ in } \Omega \\ \nabla u \cdot \vec{n} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Comparison, continued

$N \backslash \alpha$	1.0	0.01	0.0001
8	21	22	21
16	17	20	23
32	31	29	25
64	24	26	27
128	23	24	30
256	35	34	34
512	32	34	37

(a) Schöberl and Zulehner

$N \backslash \alpha$	1.0	0.01	0.0001
8	17	17	17
16	15	15	15
32	31	26	27
64	23	23	23
128	23	21	21
256	33	41	32
512	32	31	30

(b) our approach