Finding the set of measurements necessary to reveal the state of a static system

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Abstract
A static system whose state is a linear combination of a finite set of ground states is considered. Based on the measurement of a set of characteristics we want to make an educated guess of the state of the system. The measurements can be expensive and/or time-consuming, and we would like to avoid making unnecessary ones. So we want to determine which characteristics should be measured. Several approaches to this problem are described. All of them are based on integer linear programming.

1. Introduction

We consider a static system whose state s is a linear combination of G ground states. We cannot measure the state, which is a finite-dimensional vector, directly, but we can measure up to M measurable characteristics of the system. Based on the values e of these measurements we want to make an educated guess of the system's state s.

We assume that we have been able to establish an $M \times G$ -dimensional matrix F where column g is a reasonable estimate of what the M characteristics would be if the system is in ground state g.

The investigations which motivated this research are concerned with the detection of a patient's heart condition by registering the potentials measured by electrodes placed on different locations on a patient's torso. So in this case the characteristics are the electrode potentials whilst the state is the condition of the heart.

2 Quality of the matrix *F*. Selecting a good set of measurements

A measure of the ability of the matrix F to differentiate between the different ground states is $\min_{1 \le g < g' \le G} \sum_{m=1}^{M} | f_{mg} - f_{mg'} |$.

To make a measurement can be costly and/or time-consuming. We would therefore like to avoid making unnecessary ones. Therefore we would like for each number n between 2 and $M-1$ to establish the best submatrix of F to use when we limit ourselves to measuring n characteristics. This we achieve through solving the following integer programming problem:

(2.1) $\max u$ subject to

(2.2) $\sum_{m=1}^{M} | f_{mg} - f_{mg'} | x_m \geq u$ for $1 \leq g < g' \leq G$

(2.3) $\sum_{m=1}^{M} x_m = n$

where the variables are

$$x_m = \begin{cases} 1 \text{ if measurement } m \text{ shall be made} \\ 0 \text{ otherwise} \end{cases}$$

u: lower bound on column separation

Based on how max u varies with n we should be able to select a good subset of measurements to be used.

3. Solution of the estimation problem

When we measure the column e of characteristics of the system we cannot expect e to coincide with any particular linear combination s of the columns of F. There is too much noise. So, once we have decided on a good set of measurements e, we find the column vector s which minimizes $(e - Fs)^2$ or, if the system $e = Fs$ has solutions, the solution with minimum Euclidean norm, and make s our guess of the system's state. The solution to the minimization problem is

(3.1) $s = F^{\mathrm{I}} e$

where F^{I} is the generalized inverse of F.

4 Quality of the matrix F^I

From (3.1) we see that an estimate of the sensitivity of ground state component g to variations in the value e_m of the characteristic m is $| f_{gm}^{\mathrm{I}} | = | \partial s_g / \partial e_m |$. For each ground state g there should be at least one characteristic m such that $| f_{gm}^{\mathrm{I}} |$ is high. Let us for each ground state g pick a characteristic $m(g)$ which secures the highest sensitivity $| f_{gm(g)}^{\mathrm{I}} |$. We want both the smallest and the average of these sensitivities to be large. So a measure of quality could be a balance between these two quantities:

(4.1) $\lambda \min_{1 \leq g \leq G} | f_{gm(g)}^{\mathrm{I}} | + (1 - \lambda) \sum_{g=1}^{G} | f_{gm(g)}^{\mathrm{I}} | / G$

where λ is a balancing parameter between 0 and 1.

For each of the relevant submatrices of F found in Section 2 we can calculate the expression (4.1) and thus obtain more information to consider when deciding which submatrix to use.

5 Selecting a good set of measurements based on the quality of F^{I}

As a supplement to the approach described in Section 2 we can for each number n between 2 and $M-1$ establish the best solution quality we can obtain when we limit ourselves to measuring n characteristics.

Ideally we should then for each of the 2^{M-1} subsets N of M focus on the corresponding submatrix F_N of F, establish the generalized inverses F_N^{I} and calculate the quality of F_N^{I} using (4.1). This is deemed to be infeasible. Instead we stick to the full matrix F^{I} and for each n solve the following mixed integer programming problem:

(5.1) $$\max\ z = \lambda y + (1-\lambda)\frac{1}{G}\sum_{\substack{1\le m\le M\\ 1\le g\le G}} |f_{gm}^{\mathrm{I}}|\, x_{mg} \quad \text{subject to}$$

(5.2) $$\sum_{m=1}^{M} x_{mg} = 1$$

(5.3) $$\sum_{m=1}^{M} |f_{gm}^{\mathrm{I}}|\, x_{mg} \ge y$$

(5.4) $$x_{mg} \le x_m$$

(5.5) $$\sum_{m=1}^{M} x_m = n$$

where the variables are

$$x_{mg} = \begin{cases} 1 \text{ if measurement } m \text{ is chosen to secure that the sensitivity} \\ \text{towards ground state component } g \text{ is above a threshold } y \\ 0 \text{ otherwise} \end{cases}$$

x_{mg} is required to be binary,

$$x_m = \begin{cases} 1 \text{ if measurement } m \text{ shall be made} \\ 0 \text{ otherwise} \end{cases}$$

x_m is not explicitly required to be binary, but required to lie between 0 and 1

y which is a lower threshold for achieved sensitivity.

Solving this optimization problem will give the solution quality as a function of n. This information represents a supplement to the information obtained by solving (2.1) – (2.3).

Inspired by stepwise regression we may also run the optimization above for $n = M - 1$, register the solution quality, recalculate F^I based on the reduced matrix *F*, run the optimization for $n = M - 2$, register the new solution quality and so on.

6 An alternative approach

Here we shall restrict ourselves to cases where *F* has rank *G*. Then the generalized inverse of *F* takes the form

(6.1) $$F^{\mathrm{I}} = (F^{\mathrm{T}}F)^{-1}F^{\mathrm{T}}$$

In some applications the columns of the matrix *F* are very close, relatively speaking, so it is rather difficult to differentiate between different states *s* based directly on a subset of the *M* measurements. In mathematical terms, the matrix *F* is ill-conditioned.

In lieu of using the values of the measurements *e* directly one may base the estimate of *s* on a set of *C* linear combinations of the measurements. This amounts to choose a $C \times M$ matrix *Z* to give a new set of 'measurements' *Ze*. We have that

(6.2) $$Ze = ZFs = Ls \text{ where } L = ZF .$$

From (6.1) we see that the least square estimate of *s* based on *Ze* is

(6.3) $$s = ((ZF)^{\mathrm{T}} ZF)^{-1}(ZF)^{\mathrm{T}} Ze = (L^{\mathrm{T}}L)^{-1}L^{\mathrm{T}}Ze = PZe \text{ where } P = (L^{\mathrm{T}}L)^{-1}L^{\mathrm{T}} .$$

We see the difference between the least square estimators (3.1) and (6.3) more clearly by rewriting (6.3) as

(6.4) $$s = (F^{\mathrm{T}}Z^{\mathrm{T}}ZF)^{-1}F^{\mathrm{T}}Z^{\mathrm{T}}Ze.$$

From (6.4) we see that when *F* is square and nonsingular and *Z* is chosen to be nonsingular, then both (3.1) and (6.4) become $s = F^{-1}e$ independent of *Z*.

We want to choose *Z* so that the columns of the matrix $L = ZF$ are 'far' from each other. We define the *separation* between two columns to be the sum of the absolute values of the differences between corresponding column entries. We want to find a transformation matrix Z so that the minimum column separation in *L* is as large as possible. In order to have a meaningful comparison with the minimum column separation in *F* we require that the sum of the absolute values of the elements in *L* is equal to C / M times the sum of the absolute values of the elements in *F*. Furthermore, since there may be uncertainties associated with the numbers in *F*, we do not want to blow up such uncertainties by allowing the absolute values of the elements in *Z* to be too large. Finding the best *Z* can be formulated as follows:

(6.5) max y subject to

(6.6) $|(ZF)_g - (ZF)_{g'}| \geq y$ for $1 \leq g < g' \leq G$

(6.7) $|z_{cm}| \leq Bx_m$ for $1 \leq c \leq C, 1 \leq m \leq M$

(6.8) $\sum_{m=1}^{M} x_m = n$

(6.9) $\sum_{\substack{1 \leq c \leq C \\ 1 \leq g \leq G}} |(ZF)_{cg}| = \frac{C}{M} \sum_{\substack{1 \leq m \leq M \\ 1 \leq g \leq G}} |f_{mg}|$

where B is an appropriately chosen constant, $(ZF)_g$ is column g in ZF, and

$$x_m = \begin{cases} 1 \text{ if measurement } m \text{ is made} \\ 0 \text{ otherwise} \end{cases}$$

This is a so-called absolute value integer programming problem which can be given the following integer programming formulation:

(6.10) max y subject to

(6.11) $\sum_{m=1}^{M} z_{cm} f_{mg} = l_{cg}$ for $1 \leq c \leq C, 1 \leq g \leq G$

(6.12) $l_{cg} - l_{cg'} \geq -A \sum_{i=1}^{p-1} w_{cgi} - A \sum_{i=p}^{G} w_{cg'i} + y_{c,\min(g,g'),\max(g,g')}$

for $1 \leq c \leq C, 1 \leq g \leq G, 1 \leq g' \leq G, g \neq g, 2 \leq p \leq G$

(6.13) $\sum_{c=1}^{C} y_{cgg'} \geq y$ for $1 \leq g < g' \leq G$

(6.14) $\sum_{i=1}^{G} w_{cgi} = 1$ for $1 \leq c \leq C, 1 \leq g \leq G$

(6.15) $\sum_{g=1}^{G} w_{cgi} = 1$ for $1 \leq c \leq C, 1 \leq i \leq G$

(6.16) $z_{cm} \leq Bx_m$ for $1 \leq c \leq C, 1 \leq m \leq M$

(6.17) $-z_{cm} \leq Bx_m$ for $1 \leq c \leq C, 1 \leq m \leq M$

(6.18) $\sum_{m=1}^{M} x_m = n$

(6.19) $\sum_{\substack{1 \leq c \leq C \\ 1 \leq g \leq G}} (l_{cg}^{+} + l_{cg}^{-}) = \frac{C}{M} \sum_{\substack{1 \leq m \leq M \\ 1 \leq g \leq G}} |f_{mg}|$

(6.20) $l_{cg} = l_{cg}^{+} - l_{cg}^{-}$ for $1 \leq c \leq C, 1 \leq g \leq G$

where the variables are

z_{cm} for $1 \le c \le C, 1 \le m \le M$ (the transformation matrix)
l_{cg} for $1 \le c \le C, 1 \le g \le G$, $-A/2 \le l_{cg} \le A/2$, (the transformed F matrix)
l_{cg}^{+} for $1 \le c \le C, 1 \le g \le G$ (the positive part of l_{cg})
l_{cg}^{-} for $1 \le c \le C, 1 \le g \le G$ (the negative part of l_{cg})
$y_{cgg'}$ for $1 \le c \le C, 1 \le g < g' \le G$ (lower bound on $|l_{cg} - l_{cg'}|$)
y (lower bound on all sums $\sum_{c=1}^{C} |l_{cg} - l_{cg'}|$)

$$w_{cgi} = \begin{cases} 1 \text{ if } l_{cg} \text{ is the } i\text{-th smallest element in the } c\text{-th row of } L \\ 0 \text{ otherwise} \end{cases}$$
for $1 \le c \le C, 1 \le g \le G, 1 \le i \le G$

$$x_m = \begin{cases} 1 \text{ if measurement } m \text{ is made} \\ 0 \text{ otherwise} \end{cases} \qquad 1 \le m \le M$$

and where A and B are appropriately chosen constants.

We can reduce the density of the coefficient matrix by introducing the non-negative slack variables $s_{cgg'p}$ for $1 \le c \le C, 1 \le g \le G, 1 \le g' \le G, g \ne g', 2 \le p \le G$ into (6.12).
The constraints (6.12) for p and $p+1$ can then be written as

$$(6.21) \quad l_{cg} - l_{cg'} = -A\sum_{i=1}^{p-1} w_{cgi} - A\sum_{i=p}^{G} w_{cg'i} + y_{c,\min(g,g'),\max(g,g')} + s_{cgg'p}$$

$$(6.22) \quad l_{cg} - l_{cg'} = -A\sum_{i=1}^{p} w_{cgi} - A\sum_{i=p+1}^{G} w_{cg'i} + y_{c,\min(g,g'),\max(g,g')} + s_{cgg',p+1}.$$

Subtracting (6.22) from (6.21) gives

$$(6.23) \quad Aw_{cgp} - Aw_{cg'p} + s_{cgg'p} - s_{cgg',p+1} = 0$$
$$\text{for } 1 \le c \le C, 1 \le g \le G, 1 \le g' \le G, g \ne g', 2 \le p \le G-1.$$

So we can replace (6.12) by (6.23) and (6.21) for $1 \le c \le C,, 1 \le g \le G, 1 \le g' \le G, g \ne g'$ and $p = 2$:

$$(6.24) \quad l_{cg} - l_{cg'} = -Aw_{cg1} - A\sum_{i=2}^{G} w_{cg'i} + y_{c,\min(g,g'),\max(g,g')} + s_{cgg'2}.$$

Still, this formulation is difficult to solve. Therefore we resort to the following heuristic. We start by drawing C random permutations ($p_c(1), \cdots, p_c(G)$) of the numbers $1, \cdots, G$. The interpretation of these permutations is that if $p_c(g) < p_c(g')$ then $l_{cg} \le l_{cg'}$. The maximization of the minimal column separation can then be achieved by solving the problem

(6.25) max y subject to

(6.26) $\sum_{m=1}^{M} z_{cm} f_{mg} = l_{cg}$

(6.27) $\sum_{c=1}^{C} sign(p_c(g') - p_c(g))(l_{cg'} - l_{cg}) \geq y$ for $1 \leq g < g' \leq G$

(6.28) $z_{cm} \leq Bx_m$ where B is a large constant

(6.29) $-z_{cm} \leq Bx_m$

(6.30) $l_{cg} = l_{cg}^{+} - l_{cg}^{-}$

(6.31) $\sum_{\substack{1 \leq c \leq C \\ 1 \leq g \leq G}} (l_{cg}^{+} + l_{cg}^{-}) = \frac{C}{M} \sum_{\substack{1 \leq m \leq M \\ 1 \leq g \leq G}} | f_{mg} |$

(6.32) $\sum_{m=1}^{M} x_m = n$

(6.33) $sign(p_c(g') - p_c(g))(l_{cg'} - l_{cg}) \geq 0$ for $1 \leq c \leq C, 1 \leq g < g' \leq G$

where the variables are

z_{cm} (elements of the transformation matrix)

l_{cg} (the elements in the transformed F matrix)

l_{cg}^{+} (the positive part of l_{cg})

l_{cg}^{-} (the negative part of l_{cg})

y (a lower bound on all sums $\sum_{c=1}^{C} | l_{cg} - l_{cg'} |$)

$$x_m = \begin{cases} 1 \text{ if measurement } m \text{ is made} \\ 0 \text{ otherwise} \end{cases}$$

We use a depth first heuristic for securing binary values of the x-variables by requiring them to lie between 0 and 1, and successively rounding up or down a variable that is closest to being integer.

7 Example

Our example is a 72×17 matrix F is shown in Table 6 where the entries are scaled up by a factor 1000. It originates from a case where the states are heart conditions and the measurements are electrode potentials as mentioned in the introduction. We shall use both approaches. For solving the integer programming problems we have used the GNU linear programming kit. The transpose of the corresponding matrix F^{I} is shown in Table 7. The entries here are scaled up by a factor 10 and rounded. We see that the entries in F^{I} are two orders of magnitude greater than the entries in F. This, together with that the condition number (the ratio between the largest and the smallest singular value) is 2016, indicate that F^{I} is ill-conditioned.

7.1 Using the first approach

The results of solving (2.1) – (2.3) for $n = 1, \cdots, 72$ are shown in Table 1.

We see, for example, that if we want the drop in minimum separation by removing one measurement to be ≤ 0.01 we should make at least 15 measurements.

Running the optimization (5.1) – (5.5) for the matrix in Table 7 with $\lambda = 0.75$ for different *n* gives the results shown in Table 2.

No of measurements	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Separation * 1000	2	24	46	69	82	104	116	129	148	158	175	189	199	211	220	229	238	245
No of measurements	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
Separation * 1000	253	263	270	276	284	292	298	306	312	319	326	332	338	345	351	357	364	370
No of measurements	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
Separation * 1000	376	382	388	393	399	405	411	416	421	426	432	436	442	447	452	456	461	465
No of measurements	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
Separation * 1000	469	474	477	480	483	485	487	489	490	492	492	493	493	494	494	495	495	495

Table 1. Optimal solutions of (2.1) – (2.3)

We see that we do not need more than 5 measurements in order to obtain the best possible quality. This indicates that it is probably not wise to base the measurement selection exclusively on the quality measure described in Section 4.

n	Quality	Measurements used
1	34.1	13
2	36.2	13 20
3	38.2	13 14 20
4	38.7	12 13 14 20
5	38.8	5 12 13 14 20
72	38.8	1 - 72

Table 2. Results from running (5.1) – (5.5)

7.2 Using the alternative approach

We select $C = 72$ and use the heuristic described in Section 6. The sum of the absolute values of the elements in *F* is 30.9 which then becomes the right hand side of (6.31).

To explore the sensitivity of the suboptimal separation to variations in *B* we first solve the problem (6.25) – (6.33) with all 72 measurements for B = 50, 40, 30, 20, 10, 5 and 1. The results are shown in Table 3.

Then we do the same for 10 measurements with results shown in Table 4.
We see from these results that the separations obtained are not too sensitive to variations in *B* for $B \geq 5$. We have kept *B* at 5 in the runs which follow. Of course, if the uncertainties in the entries in *F* are feared to be high, it may be prudent to set *B* to 1.

B	Separation
50	3.64
40	3.64
30	3.64
20	3.60
10	3.42
5	3.10
1	1.88

Table 3. Separation obtained for different B-s. 72 measurements

We now use start by optimizing using one characteristic only. Then we optimize for two characteristics where we keep the characteristic found in the previous run. We continue in this way until we reach all 72 characteristics. The results of these runs are shown in Table 5.

B	Separation
50	1.93
40	1.80
30	1.80
20	1.80
10	1.48
5	1.46[1]
1	1.06

Table 4. Separation obtained for different B-s. 10 measurements

No of measurements	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Separation * 100	2	19	33	53	76	100	110	132	142	155	169	181	190	198	204	209	214	222
Added characteristic	22	3	20	7	5	12	65	6	67	14	4	55	11	62	15	66	48	21
No of measurements	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
Separation * 100	227	231	235	240	256	259	262	265	268	270	272	274	276	278	280	282	284	285
Added characteristic	53	1	61	23	13	68	18	8	2	24	47	19	63	60	42	58	28	41
No of measurements	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
Separation * 100	286	288	289	290	291	293	294	295	296	297	298	298	299	300	301	302	302	303
Added characteristic	10	30	56	32	54	27	16	29	46	59	64	9	44	26	40	57	52	17
No of measurements	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
Separation *100	304	304	305	305	306	306	307	307	307	308	308	308	309	309	309	309	310	310
Added characteristic	31	39	43	25	45	34	70	33	72	38	49	69	35	37	50	36	71	37

Table 5. Separation and chosen characteristics

We see that, even if we only have obtained suboptimal solutions to (6.25) – (6.33), we are able to obtain transformations Z which improve column separation almost an order of magnitude above the separations listed in Table 1.

[1] The difference between this number and the corresponding number in Table 5 is a consequence of suboptimization.

To get a feeling for the influence of the permutations chosen we make a series of runs with all 72 measurements made where we generate the randomly chosen permutations at random. The results are shown in Table 6.

Separation	3.03	3.04	3.05	3.06	3.07	3.08	3.09	3.10	3.11	3.12	3.13	3.14
Frequency	1	0	0	3	4	5	12	5	9	5	3	3

Table 6. Results of varying the permutations randomly. 72 measurements.

Finally, to get a feeling for how much worse we do with the heuristic (6.105) – (6.20) in lieu of solving (6.25) – (6.33) when we stop at the first integer solution obtained, we solved both problems using the upper left 18×4 corner of the F matrix. Both methods gave the same minimal column separation 0.748 albeit with different matrices L.

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-235	-261	-256	-36	227	246	-201	-220	-99	39	269	320	31	136	274	52	863
-269	-381	-312	-28	272	282	-265	-340	-182	13	306	362	-24	112	279	18	874
-220	-701	-482	-37	388	402	-374	-741	-539	-132	400	500	-245	-27	239	-100	804
117	-898	-572	-66	508	573	-252	-1308	-1189	-484	480	723	-678	-425	18	-267	360
704	-456	-180	-15	508	670	448	-1373	-1601	-1054	329	947	-1468	-1480	-971	-158	-1883
767	134	115	-85	92	318	725	-182	-799	-909	-362	406	-759	-1171	-1882	239	-3839
565	248	135	-112	-157	7	531	164	-317	-485	-482	-83	-224	-536	-1129	98	-2464
487	276	96	-168	-231	-71	459	241	-218	-390	-451	-161	-66	-341	-777	89	-1524
-345	-266	-181	42	214	187	-280	-190	5	118	258	248	60	173	309	24	877
-424	-475	-263	64	287	239	-409	-388	-112	92	320	307	-26	141	323	-37	903
-486	-839	-367	115	390	305	-632	-788	-348	43	409	373	-219	50	330	-191	874
-63	-1237	-375	225	584	503	-674	-1891	-1089	-213	578	580	-1000	-494	141	-669	187
680	-421	161	376	550	541	359	-1446	-924	-386	461	732	-2291	-1601	-811	-539	-4414
607	159	269	143	32	80	541	-18	-250	-323	-305	-31	-598	-782	-1353	-37	-5088
507	239	188	-26	-154	-62	465	180	-180	-342	-451	-212	-202	-457	-1000	6	-2713
420	260	155	-78	-225	-141	387	248	-95	-273	-431	-285	-49	-278	-678	5	-1661
-415	-152	5	142	112	10	-310	-32	216	231	162	35	137	228	324	-10	781
-524	-251	20	190	132	-16	-417	-95	236	266	178	0	117	240	353	-50	800
-696	-443	75	281	144	-110	-623	-227	282	330	181	-129	63	254	396	-153	772
-659	-463	190	350	82	-274	-672	-265	354	375	101	-383	-9	248	394	-266	519
-266	-139	299	312	-77	-383	-328	-12	380	303	-107	-622	-19	168	243	-297	-298
45	87	293	198	-165	-347	-16	164	278	140	-254	-614	-5	31	-28	-248	-1094
162	188	237	78	-217	-304	130	247	182	10	-319	-509	50	-29	-201	-144	-1009
224	221	207	23	-230	-268	198	265	121	-57	-344	-445	52	-72	-296	-92	-1036
-360	-82	50	141	60	-43	-263	33	251	235	112	-33	155	225	302	-13	699
-411	-127	65	167	65	-66	-313	7	266	253	114	-63	146	231	315	-37	696
-468	-183	110	213	49	-136	-383	-21	303	283	91	-161	128	239	325	-87	631
-450	-191	181	253	-2	-243	-406	-20	346	303	24	-324	100	231	312	-156	440
-265	-54	243	231	-102	-331	-260	89	354	258	-102	-478	97	185	218	-187	50
-35	85	265	172	-176	-352	-64	184	299	159	-224	-556	71	92	48	-196	-529
97	177	229	82	-217	-316	77	251	220	54	-281	-492	91	29	-98	-130	-656
133	202	211	48	-225	-296	118	267	190	19	-291	-453	98	10	-142	-99	-649
-290	-21	35	107	53	-18	-196	70	235	213	111	3	168	219	286	17	704
-271	-2	38	101	46	-19	-178	85	236	208	105	3	173	218	280	23	698
-227	46	55	91	18	-39	-139	125	248	204	80	-20	185	214	264	30	661
-184	90	79	82	-20	-74	-103	166	261	197	43	-60	196	208	240	30	600
-123	139	104	66	-70	-120	-54	212	268	179	-8	-115	205	193	202	27	502
-50	188	126	41	-126	-168	4	257	262	147	-70	-179	207	168	148	21	369
20	218	141	15	-177	-214	56	285	241	103	-135	-243	198	133	79	10	195
59	224	154	6	-205	-244	82	292	223	71	-183	-298	182	103	24	-8	34
-278	-26	2	87	78	27	-184	54	206	199	137	59	162	216	290	35	751
-245	5	-3	72	72	36	-152	72	202	191	135	72	170	214	284	50	754
-212	50	21	71	46	12	-122	115	223	194	111	46	185	214	272	54	724
-126	141	43	45	-1	-12	-45	189	240	184	74	24	211	210	245	76	680
-63	195	87	32	-74	-90	3	248	257	166	0	-68	221	191	197	63	546
45	260	94	-21	-144	-136	91	304	231	112	-69	-118	225	157	125	72	410
112	272	108	-54	-203	-193	139	320	194	50	-152	-200	205	108	33	53	188
150	269	118	-65	-231	-220	165	319	162	5	-207	-254	182	67	-39	34	2
-213	-76	-123	4	144	151	-140	-29	71	120	200	210	122	183	278	75	828
-155	-12	-112	-17	121	146	-85	21	82	114	183	208	140	182	265	96	814
-106	43	-103	-33	98	140	-40	67	94	112	166	203	157	182	253	112	800
0	143	-92	-77	47	122	53	149	101	95	128	190	184	173	219	144	757
68	239	-34	-85	-34	45	115	246	148	99	59	109	218	168	182	149	665
211	304	-50	-184	-136	-9	231	309	79	4	-37	55	213	107	70	168	477
273	320	-20	-214	-214	-82	281	333	39	-71	-142	-42	193	46	-45	149	225
290	313	21	-201	-251	-134	292	333	23	-114	-217	-127	168	2	-133	117	-4
-157	-105	-191	-53	165	203	-105	-75	-15	62	218	274	92	156	258	91	833
-111	-55	-184	-72	146	199	-60	-35	-8	56	204	272	107	155	247	109	821
-52	8	-176	-97	119	193	-6	18	2	50	184	267	126	153	231	130	803
57	87	-189	-160	75	190	87	82	-20	14	151	271	142	135	190	164	759
185	177	-199	-244	-7	159	199	164	-56	-43	82	244	157	99	117	198	653
286	271	-160	-291	-119	75	290	261	-59	-90	-23	153	175	59	23	208	473
345	313	-90	-298	-218	-24	341	311	-72	-158	-153	25	162	-5	-111	184	173
350	316	-32	-266	-253	-82	345	320	-62	-181	-223	-62	148	-39	-187	152	-34
444	70	-470	-646	2	341	402	4	-542	-446	57	468	9	-80	-71	299	546
491	246	-295	-549	-182	167	463	201	-411	-406	-137	258	68	-122	-223	282	196
495	293	-157	-446	-254	56	473	258	-339	-410	-267	104	60	-176	-366	237	-168
450	305	-32	-312	-273	-47	432	288	-214	-344	-333	-50	63	-180	-429	167	-472
-233	42	135	130	-56	-167	-162	151	298	217	-9	-194	179	205	231	-33	463
-129	114	172	106	-130	-235	-81	214	297	179	-96	-290	181	174	160	-48	254
-55	153	184	82	-170	-264	-24	245	279	142	-149	-337	174	143	100	-55	94
19	189	183	50	-199	-274	37	270	248	97	-191	-358	166	109	35	-50	-50

Table 6. The matrix $10000 \times F$

-258	-145	-116	7	84	51	-236	-73	77	152	179	94	164	250	405	6	1193
-265	-182	-133	12	110	76	-253	-128	41	142	205	126	128	230	408	-5	1197
-222	-288	-178	20	188	166	-254	-318	-122	61	264	241	-42	98	330	-43	970
12	-384	-210	8	279	312	-109	-609	-472	-178	274	416	-426	-274	-30	-96	-31
845	-235	-47	-33	257	444	561	-876	-1125	-817	-39	521	-1280	-1281	-1314	-163	-3692
1313	198	243	-61	-12	231	1047	-469	-1068	-1025	-482	190	-1381	-1614	-2050	-115	-5845
818	227	205	-44	-85	58	684	-127	-541	-593	-377	4	-741	-929	-1267	-45	-3619
550	197	146	-46	-89	12	473	-12	-318	-382	-277	-29	-432	-580	-830	-9	-2334
-296	-141	-105	19	76	29	-264	-46	125	188	179	65	203	294	450	3	1293
-316	-205	-131	31	118	66	-300	-134	75	179	224	114	151	271	468	-16	1327
-323	-315	-169	54	192	135	-341	-302	-37	142	290	201	26	192	452	-57	1248
-42	-502	-208	76	347	340	-193	-767	-521	-166	335	444	-542	-326	-15	-166	-136
1301	-150	175	60	228	429	921	-1029	-1406	-1128	-277	436	-1858	-1917	-2142	-285	-6388
1393	290	386	6	-82	124	1130	-376	-989	-1028	-613	22	-1446	-1710	-2251	-157	-6642
822	240	236	-24	-98	33	688	-109	-516	-583	-399	-34	-749	-939	-1294	-58	-3755
535	204	168	-28	-98	-10	462	5	-284	-361	-289	-63	-423	-568	-825	-18	-2370
-307	-84	-60	31	31	-30	-256	48	212	226	133	-17	266	338	456	9	1270
-340	-108	-63	45	43	-30	-289	30	223	246	150	-16	271	357	493	0	1348
-385	-145	-60	72	59	-35	-337	-1	238	273	171	-24	265	374	533	-22	1405
-341	-121	-20	88	41	-59	-299	8	235	252	129	-65	222	320	447	-37	1109
-73	23	78	65	-38	-94	-55	77	140	90	-39	-134	61	71	50	-33	-27
205	129	148	35	-85	-85	188	79	-17	-101	-178	-145	-148	-215	-364	-31	-1180
249	157	139	8	-97	-75	231	98	-43	-134	-195	-130	-149	-237	-405	-12	-1234
288	167	137	-3	-98	-64	265	91	-76	-165	-206	-115	-176	-275	-455	-7	-1349
-279	-54	-42	28	11	-47	-226	79	220	215	104	-40	266	323	416	13	1154
-293	-63	-41	35	15	-49	-240	73	226	224	109	-44	268	330	430	8	1178
-298	-65	-29	46	13	-58	-245	72	236	231	104	-59	263	328	425	0	1136
-261	-43	0	55	-2	-76	-213	84	233	212	71	-89	233	287	357	-8	911
-132	27	49	46	-42	-96	-95	120	190	135	-13	-126	156	167	164	-7	357
62	102	104	30	-77	-96	73	121	83	3	-114	-143	6	-35	-128	-11	-474
146	135	113	9	-90	-85	147	125	30	-57	-150	-133	-44	-112	-240	-1	-754
162	141	110	2	-92	-80	162	125	18	-70	-154	-126	-51	-125	-258	2	-788
-262	-42	-42	19	6	-45	-209	88	213	205	96	-38	266	314	400	20	1125
-256	-36	-41	17	3	-46	-203	94	213	202	92	-40	266	312	393	22	1109
-238	-18	-32	13	-7	-54	-184	111	215	194	76	-50	266	302	369	25	1047
-215	1	-20	10	-20	-63	-160	130	216	183	56	-64	262	286	336	28	953
-178	28	-4	6	-36	-72	-124	149	209	163	28	-80	247	257	282	31	802
-128	57	14	0	-53	-80	-78	165	193	133	-4	-95	221	213	209	33	595
-66	83	34	-3	-67	-84	-24	171	164	92	-40	-106	177	151	116	32	331
-16	100	50	-4	-75	-85	17	168	136	58	-67	-112	134	95	37	28	101
-267	-51	-53	16	14	-35	-215	76	204	204	107	-23	265	317	411	21	1170
-260	-45	-53	12	10	-35	-208	83	203	199	103	-23	266	314	405	24	1158
-247	-28	-44	9	0	-44	-193	102	209	196	89	-36	270	310	388	28	1113
-218	2	-32	1	-18	-55	-162	132	213	184	64	-51	274	298	353	36	1025
-173	36	-9	-2	-39	-70	-117	160	210	160	27	-76	257	262	285	39	829
-111	71	9	-12	-58	-76	-60	178	186	122	-10	-88	226	209	197	43	592
-35	98	32	-16	-73	-78	4	177	144	70	-52	-98	165	127	80	39	257
24	114	50	-18	-80	-76	53	167	106	27	-81	-101	110	59	-12	34	-15
-258	-83	-83	5	40	2	-217	22	150	179	136	29	229	291	408	21	1195
-242	-64	-77	-1	29	-3	-199	42	153	172	122	21	235	287	391	27	1156
-229	-48	-72	-6	19	-9	-183	60	158	168	109	12	240	284	376	33	1122
-195	-15	-60	-17	0	-19	-147	92	158	151	83	0	242	268	335	43	1024
-160	26	-37	-22	-26	-41	-109	137	173	140	45	-33	248	253	288	50	890
-69	68	-15	-38	-48	-43	-27	151	128	80	-3	-41	195	170	160	55	550
16	102	12	-42	-66	-47	46	153	83	21	-52	-55	127	79	27	51	166
81	122	37	-40	-76	-50	100	146	46	-23	-88	-66	66	2	-80	43	-154
-239	-96	-96	-2	52	23	-206	-8	112	155	144	57	198	262	385	20	1149
-226	-81	-91	-7	44	19	-191	7	114	150	133	50	202	258	371	25	1116
-208	-61	-85	-13	32	12	-172	28	117	143	118	42	207	253	352	32	1072
-170	-34	-78	-26	18	10	-134	47	105	119	95	37	198	227	305	41	960
-107	4	-62	-41	-2	4	-74	73	84	81	57	26	175	180	222	50	750
-34	54	-33	-51	-32	-12	-4	109	69	40	4	0	147	122	117	56	460
62	100	5	-54	-58	-24	81	124	29	-20	-57	-27	78	24	-33	52	18
116	121	30	-51	-71	-32	127	125	5	-55	-92	-44	31	-37	-125	45	-259
44	-24	-105	-84	37	98	32	-58	-122	-69	59	147	-3	-12	25	50	292
139	66	-37	-84	-22	46	130	28	-103	-108	-36	66	-25	-80	-125	54	-164
230	115	13	-77	-51	22	212	49	-128	-160	-104	24	-94	-177	-280	45	-641
279	146	57	-61	-73	-3	256	68	-128	-183	-151	-19	-135	-234	-375	32	-965
-200	9	0	22	-28	-76	-148	133	218	177	37	-87	242	264	299	18	815
-127	52	29	15	-52	-89	-80	156	195	134	-11	-110	202	197	190	20	505
-69	77	46	10	-65	-92	-29	162	167	95	-44	-119	161	140	104	20	261
-12	98	60	3	-75	-90	19	162	135	56	-72	-121	119	82	21	20	31

Table 7. The matrix $10 \times (F^1)^{\mathrm{T}}$