Shape Optimization with Multiple Meshes in the FEniCS-framework

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September 28, 2017

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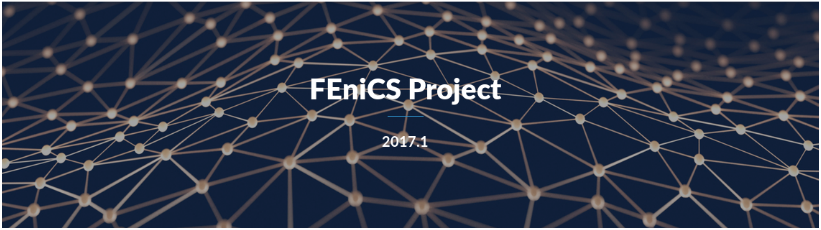


The FEniCS computing platform

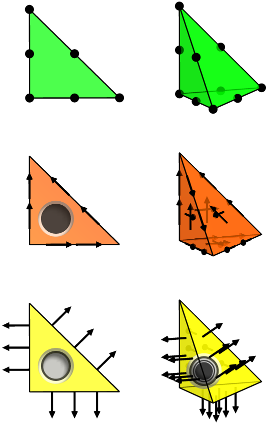
FEniCS is a popular open-source (LGPLv3) computing platform for solving partial differential equations (PDEs). FEniCS enables users to quickly translate scientific models into efficient finite element code. With the high-level Python and C++ interfaces to FEniCS, it is easy to get started, but FEniCS offers also powerful capabilities for more experienced programmers. FEniCS runs on a multitude of platforms ranging from laptops to high-performance clusters.

[fenicsproject.org]

- FEniCS is a software for solving PDEs via the finite-element method
- FEniCS is an international open source software and research project
- FEniCS is user-friendly: estimated $10^3 - 10^4$ users world-wide
- FEniCS is efficient: parallel performant up to (at least) 25 000 cores.

FEniCS provides automated generation of bases for a wide range of finite element spaces

```
from dolfin import *

# Import meshes
mesh = Mesh("cable.xml")
subdomains = MeshFunction("size_t", mesh,
                          "cable_vf.xml")

# Define finite element spaces
V = FunctionSpace(mesh, "CG", 1)
u = TrialFunction(V)
v = TestFunction(V)
T = Function(V)

# Problem specific variables
f= Expression("cos(x[0])*exp(sin(x[1]))", degree=3)
lmb = Expression("...", degree=3)
T_ex = 20.
c = 0.01

# Define variational form
a = inner(lmb*grad(u), grad(v))*dx+u*v*ds-c*u*v*dx
l = f*v*dx+T_ex*v*ds

# Solve a(T,v) = l(v) with respect to T
solve(a == l, T)
```



FEniCS provides an expressive form language close to mathematical syntax

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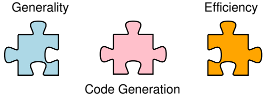
Language for variational forms



FEniCS provides automated form evaluation and assembly of the linear system

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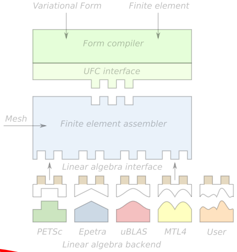
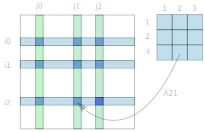
High performance linear algebra



Automated assembly



Mixed-dimensional methods[1]

[1] Cecile Daversin-Catty and Marie E. Rognes. "Automated abstractions for Mixed-Dimensional Finite Element Methods". In: *Preparation* ().

Multi-physics problems require efficent mixed-dimensional and mixed domain coupling: emerging features in FEniCS!

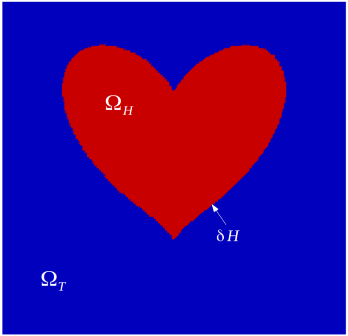
```
# W = V(Omega_H) x V(Omega_H U Omega_T)
V = FunctionSpace(mesh, "Lagrange",1) # Heart + Torso
H = FunctionSpace(submesh_heart, "Lagrange",1) # Heart
W = FunctionSpaceProduct(H,V)

# v, psi_v in V(Omega_H)
# u, psi_u in V(Omega_H U \Omega_T)
(v,u) = TrialFunction(W)
(psi_v,psi_u) = TestFunction(W)

# Integration on the heart domain Omega_H
dH = Measure("dx",domain=W.sub_space(0).mesh())
# Integration on the whole domain Omega_H U Omega_T
dV = Measure("dx",domain=W.sub_space(1).mesh())

# Variational formulation
A = v*psi_v*dH\
    + th*dt*Mi*inner(grad(v),grad(psi_v))*dH
C = (dt/th)*(Mi+Me)*inner(grad(u),grad(psi_u))*dV\
    + (dt/th)*Mt*inner(grad(u),grad(psi_u))*dV
B = dt*Mi*inner(grad(u),grad(psi_v))*dH
BT = dt*Mi*inner(grad(v),grad(psi_u))*dH
a = A + C + B + BT
L = c + d

sol = Function(W)
solve(a == L, sol)
```



[C. Daversin-Catty, cecile@simula.no]

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```

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

$\phi_H^i : V(\Omega_H)$ basis functions

$\phi_{HT}^i : V(\Omega_H \cup \Omega_T)$ basis functions

$$A_{ij} = \int_{\Omega_H} \phi_H^j \phi_H^i + \theta \Delta t \int_{\Omega_H} M_i \nabla \phi_H^j \cdot \nabla \phi_H^i$$

$$B_{ij} = \Delta t \int_{\Omega_H} M_i \nabla \phi_H^j \cdot \nabla \phi_{HT}^i$$

$$C_{ij} = \frac{\Delta t}{\theta} \int_{\Omega_H} (M_i + M_e) \nabla \phi_{HT}^j \cdot \nabla \phi_{HT}^i + \frac{\Delta t}{\theta} \int_{\Omega_T} M_T \nabla \phi_{HT}^j \cdot \nabla \phi_{HT}^i$$

[C. Daversin-Catty, cecile@simula.no]

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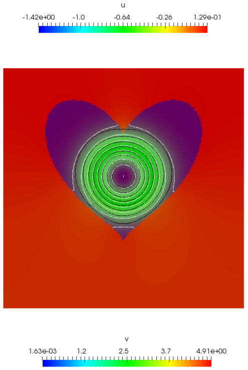

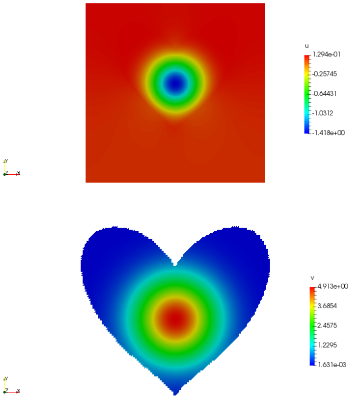

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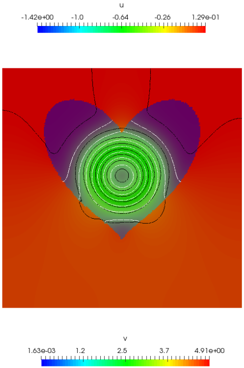

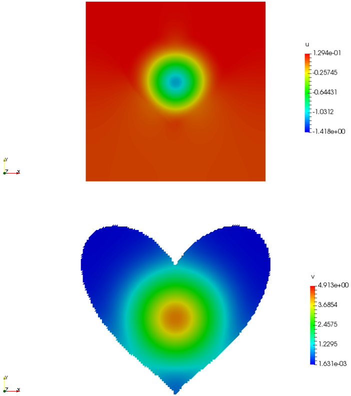

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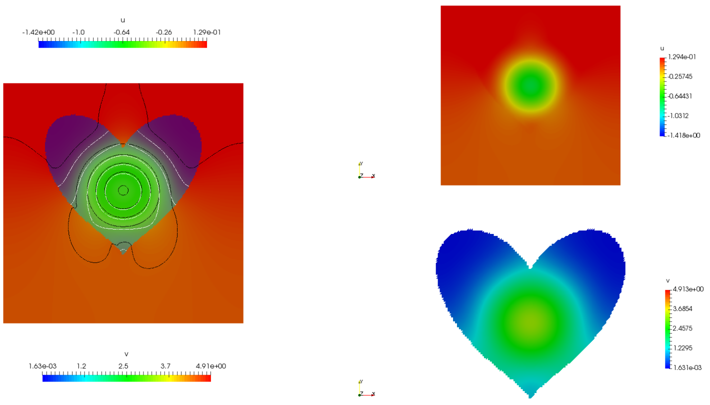

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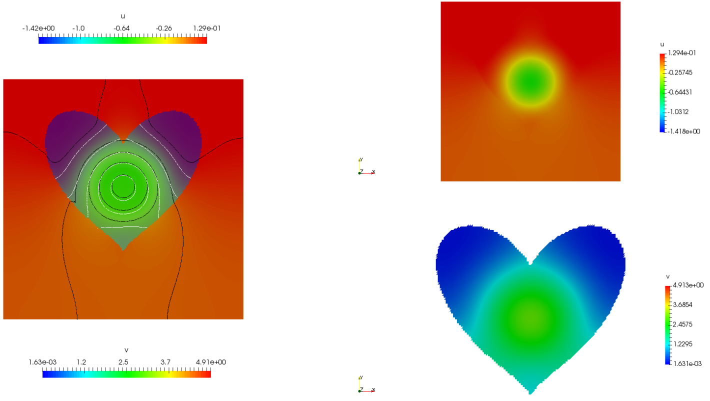
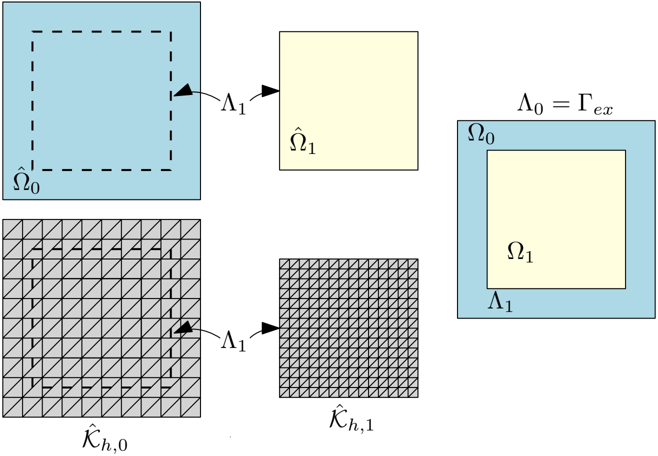


[C. Daversin-Catty, cecile@simula.no]

CUT Finite Element Methods: MultiMesh[2]

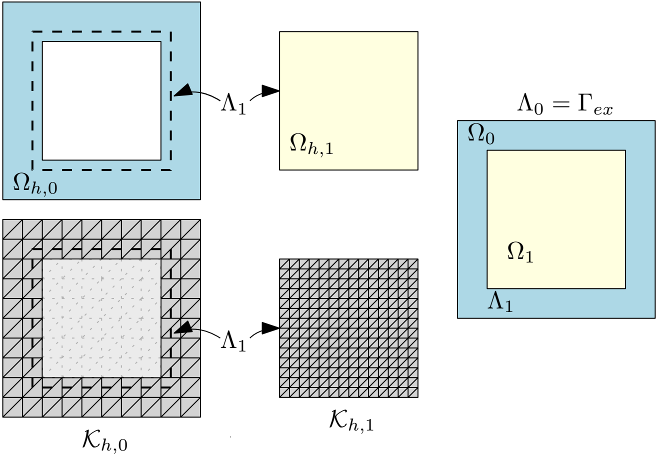
[2] August Johansson et al. "Finite Element Methods for Arbitrary Many Intersecting Meshes: Multimesh". In: *Preparation* ().

The computational domain is represented by an arbitrary number of overlapping meshes



August Johansson et al. "Finite Element Methods for Arbitrary Many Intersecting Meshes: Multimesh". In: *Preparation* ().

A finite element function space is introduced on each indvidual mesh, ignoring completely covered cells



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We illustrate the method by considering the stationary heat equation with a reaction coefficient

$$
\begin{aligned}
-\nabla \cdot (\lambda \nabla T) - cT &= f \quad \text{in } \Omega, \\
\lambda_{ex} \frac{\partial T}{\partial n} + (T - T_{ex}) &= 0 \quad \text{on } \Gamma^{ex}, \\
[T]_{\pm} &= 0 \quad \text{on } \Gamma^1_{int}, \\
\left[\lambda \frac{\partial T}{\partial n} \right]_{\pm} &= 0 \quad \text{on } \Gamma^1_{int}.
\end{aligned}
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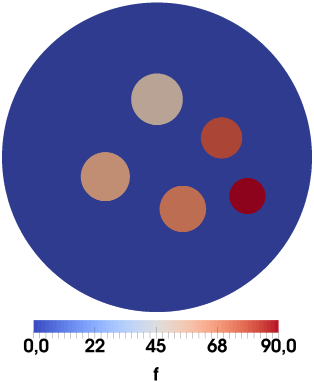

Distribution of the source f in the computational domain.

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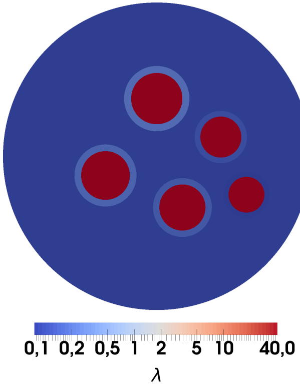

Heat diffusion coefficient λ in the computational domain.

Continuity of the solution is enforced over the artificial interface Λ_1

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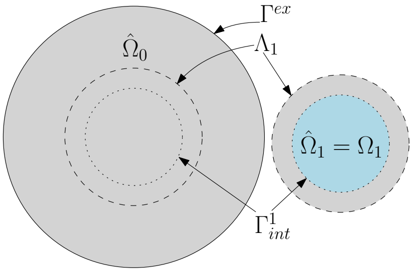

Schematic of the composition of multiple overlapping meshes.

Continuity of the solution is enforced over the artificial interface Λ_1

$$
\begin{aligned}
-\nabla \cdot (\lambda \nabla T_0) - cT_0 &= f \quad \text{in } \Omega_0, \\
-\nabla \cdot (\lambda \nabla T_1) - cT_1 &= f \quad \text{in } \Omega_1, \\
\lambda_{ex} \frac{\partial T_0}{\partial n} + (T_0 - T_{ex}) &= 0 \quad \text{on } \Gamma^{ex}, \\
[T]_{\pm} &= 0 \quad \text{on } \Gamma^1_{int}, \\
\left[\lambda \frac{\partial T}{\partial n}\right]_{\pm} &= 0 \quad \text{on } \Gamma^1_{int} \\
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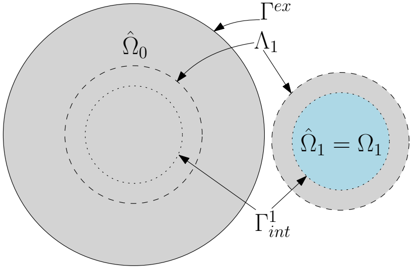


Schematic of the composition of multiple overlapping meshes.

We create a MultiCable in FEniCS by initializing the MultiMesh object and add a background mesh

```
from dolfin import *

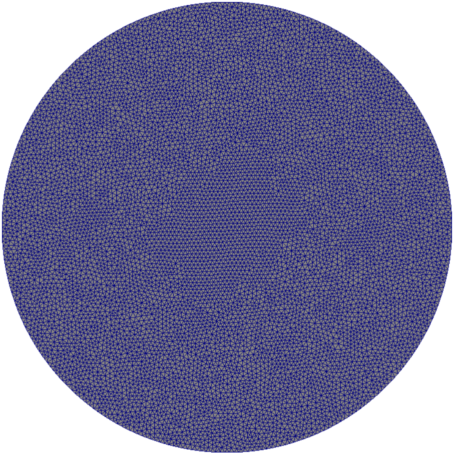
multimesh = MultiMesh()
multimesh.add(Mesh("outer_cable.xml"))
for i in range(num_cables):
    cable = Mesh("inner_cable.xml")
    # Scale and move internal cables
    # ....
    multimesh.add(cable)
multimesh.build()

# Create function space for temperature
V = MultiMeshFunctionSpace(multimesh, "CG", 1)
T = MultiMeshFunction(V, name="Temperature")
u,v = TrialFunction(V), TestFunction(V)

# Problem Specific variables
f = Expression("sin(x[0]*x[1])", degree=3)
lmb = Expression("...", degree=3)
T_ex, c = 3.2,0.04

alpha, beta = 4.0,4.0
n = FacetNormal(multimesh)
h =2.0*Circumradius(multimesh)
h = (h('+') + h('-')) / 2
F = inner(lmb*grad(u), grad(v))*dX \
            -f*v*dX -c*v*u*dX+ (u-T_ex)*v*ds
F += - inner(avg(lmb*grad(u)), jump(v, n))*dI \
     - inner(avg(lmb*grad(v)), jump(u, n))*dI \
     + alpha/h*jump(u)*jump(v)*dI  \
     + beta*inner(jump(grad(u)), jump(grad(v)))*dO

# Assemble multimesh form
A = assemble_multimesh(lhs(F))
b = assemble_multimesh(rhs(F))
solve(A, T.vector(), b, 'lu')
```



We add multiple internal cables on top of the background cable

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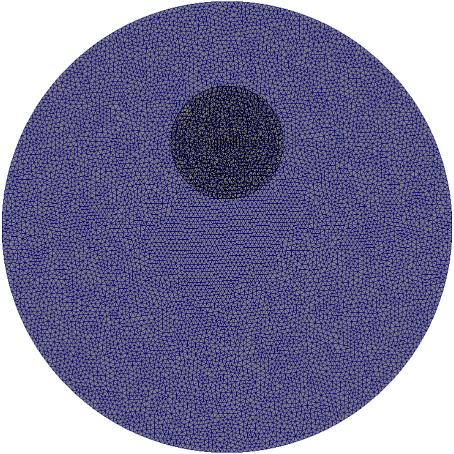

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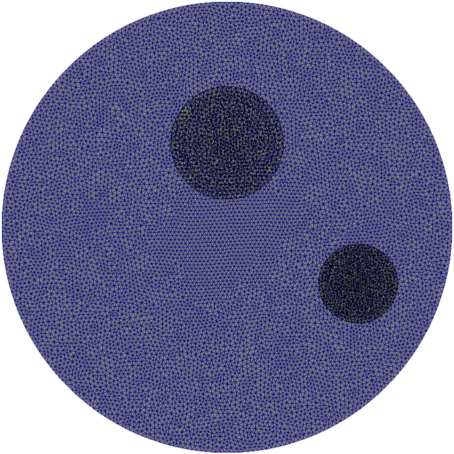

Nitsches method for weak enforcement of boundary conditions is used to obtain a stable finite element scheme

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$$0 = F_s(T, v)$$

$$F_s(T, v) = \sum_{i=0}^{1} \int_{\Omega_i} \lambda(\nabla T, \nabla v) - cTv - fv \, \mathrm{d}x + \int_{\Gamma^{\text{ex}}} (T_0 - T^{\text{ex}}) v \, \mathrm{d}s = 0$$

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A = assemble_multimesh(lhs(F))
b = assemble_multimesh(rhs(F))
solve(A, T.vector(), b, 'lu')
```

$$0 = F_s(T, v) + F_N(T, v)$$

$$F_s(T, v) = \sum_{i=0}^{1} \int_{\Omega_i} \lambda(\nabla T, \nabla v) - cTv - fv \, \mathrm{d}x + \int_{\Gamma^{\text{ex}}} (T_0 - T^{\text{ex}}) v \, \mathrm{d}s = 0$$

$$F_N(T, v) = -(\langle \lambda \mathbf{n}_1 \cdot \nabla T \rangle, [v])_{\Lambda_1} - ([T_h], \langle \lambda \mathbf{n}_1 \cdot \nabla v \rangle)_{\Lambda_1} + \frac{\beta}{h}([T], [v])_{\Lambda_1},$$

Nitsche terms	Jump	Average	
$F_N(u, v)$	$[w] = w_1 - w_0$	$\langle w \rangle = \frac{w_1 + w_0}{2}$	

Nitsches method for weak enforcement of boundary conditions is used to obtain a stable finite element scheme

```
from dolfin import *

multimesh = MultiMesh()
multimesh.add(Mesh("outer_cable.xml"))
for i in range(num_cables):
    cable = Mesh("inner_cable.xml")
    # Scale and move internal cables
    # ....
    multimesh.add(cable)
multimesh.build()

# Create function space for temperature
V = MultiMeshFunctionSpace(multimesh, "CG", 1)
T = MultiMeshFunction(V, name="Temperature")
u,v = TrialFunction(V), TestFunction(V)

# Problem Specific variables
f = Expression("sin(x[0]*x[1])", degree=3)
lmb = Expression("...", degree=3)
T_ex, c = 3.2,0.04

alpha, beta = 4.0,4.0
n = FacetNormal(multimesh)
h =2.0*Circumradius(multimesh)
h = (h('+') + h('-')) / 2
F = inner(lmb*grad(u), grad(v))*dX \
          -f*v*dX -c*v*u*dX+ (u-T_ex)*v*ds
F += - inner(avg(lmb*grad(u)), jump(v, n))*dI \
     - inner(avg(lmb*grad(v)), jump(u, n))*dI \
     + alpha/h*jump(u)*jump(v)*dI  \
     + beta*inner(jump(grad(u)), jump(grad(v)))*dO

# Assemble multimesh form
A = assemble_multimesh(lhs(F))
b = assemble_multimesh(rhs(F))
solve(A, T.vector(), b, 'lu')
```

$$0 = F_s(T, v) + F_N(T, v) + F_O(T, v)$$

$$F_s(T, v) = \sum_{i=0}^{1} \int_{\Omega_i} \lambda(\nabla T, \nabla v) - cTv - fv \,\mathrm{d}x + \int_{\Gamma^{\mathrm{ex}}} (T_0 - T^{\mathrm{ex}})v \,\mathrm{d}s = 0$$

$$F_N(T, v) = -(\langle \lambda \mathbf{n}_1 \cdot \nabla T \rangle, [v])_{\Lambda_1} - ([T_h], \langle \lambda \mathbf{n}_1 \cdot \nabla v \rangle)_{\Lambda_1} + \frac{\beta}{h}([T], [v])_{\Lambda_1},$$

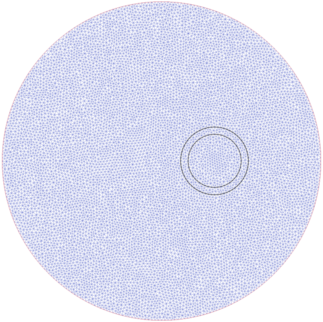
$$F_O(T, v) = ([\nabla T], [\nabla v])_{\Omega_{h,0} \cap \Omega_1}.$$

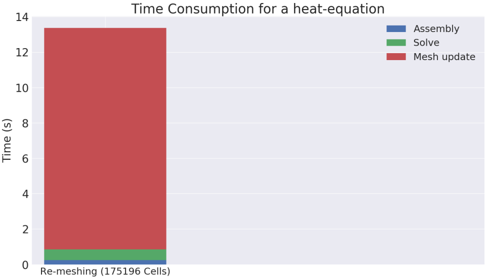
Nitsche terms	Jump	Average	Stability on overlap
$F_N(u, v)$	$[w] = w_1 - w_0$	$\langle w \rangle = \frac{w_1 + w_0}{2}$	$F_O(u, v)$

Shape-optimization With Overlapping Domains[4]

[4] Jørgen S. Dokken et al. "Shape Optimization on Multiple Meshes". In: *Preparation* ().

Re-meshing guarantees good mesh-quality, but it is a very costly operation





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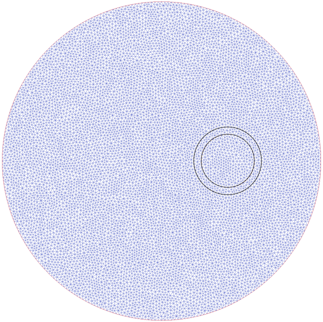

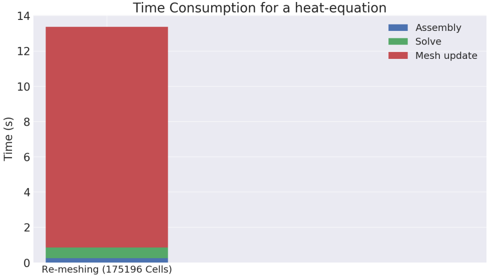

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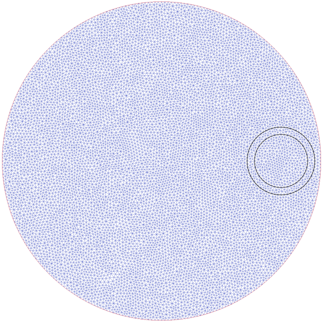
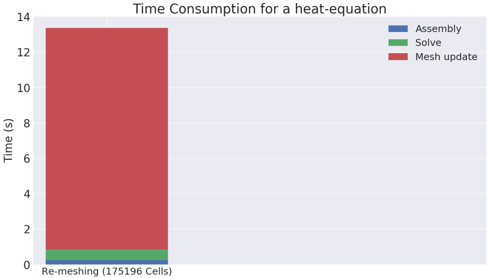
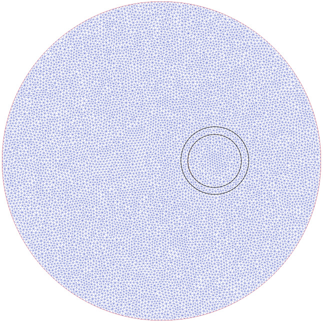
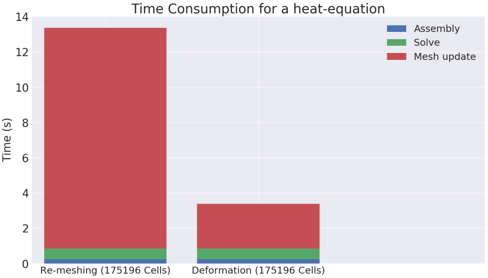




Deformation of the mesh is cheaper than re-meshing but degenerates for large changes





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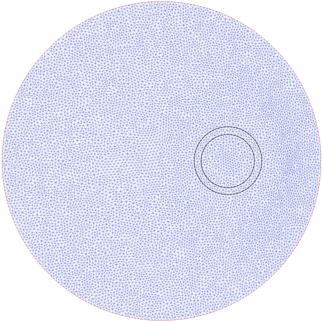

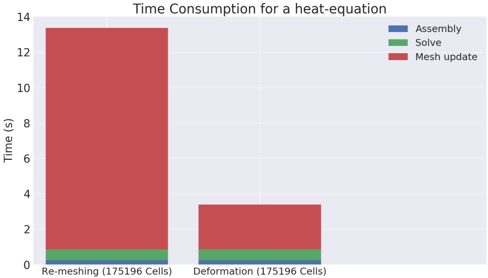

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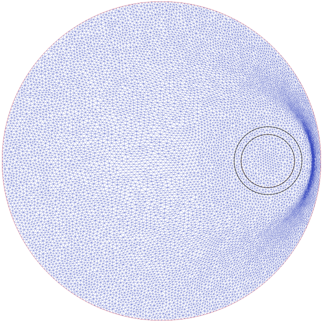
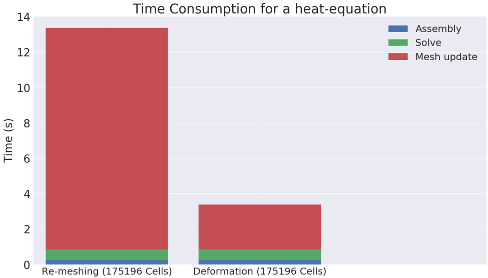
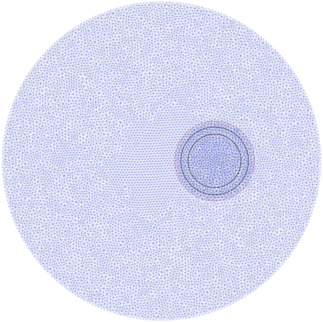
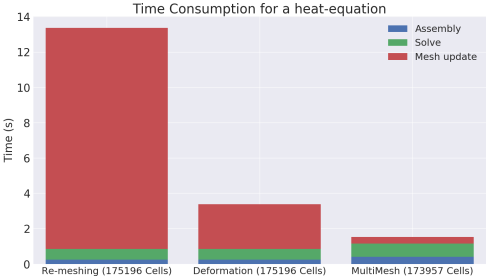




Multiple overlapping meshes is very efficient and preserves mesh quality





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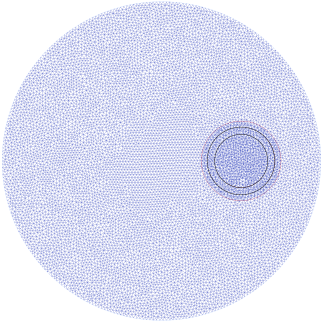

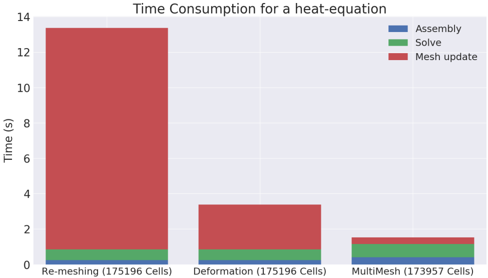

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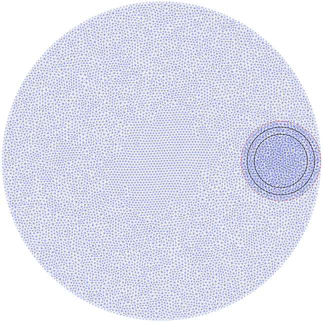

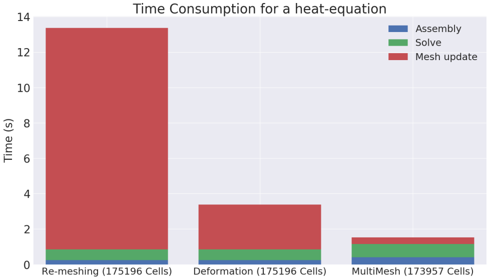

We propose a new algorithm for solving PDE-constrained shape optimization problems

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Algorithm: Shape-optimization on multiple domains

Init : **Domain composition** $\hat{\Omega}^0 = \cup_{i=0,\dots N}\hat{\Omega}_i^0$

Param: $k = 0$

while *not converged* **do**

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Result: Optimized domain

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  Increment k and set $\hat{\Omega}^k = \cup_{i=0,\dots N}\hat{\Omega}_i^k$;
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Result: Optimized domain

The solution of an optimization problem with three identical cables is an equilateral triangle

$$\min_{\Omega, T} J(\Omega, T) = \int_\Omega \frac{1}{3}|T|^3 \mathrm{d}x,$$

subject to

$$
\begin{aligned}
-\nabla \cdot (\lambda \nabla T) - cT &= f && \text{in } \Omega,\\
\frac{\partial T}{\partial n} + (T - T_{amb}) &= 0 && \text{on } \Gamma^{\text{ex}}.\\
[T]_\pm &= 0 && \text{on } \Gamma_{int},\\
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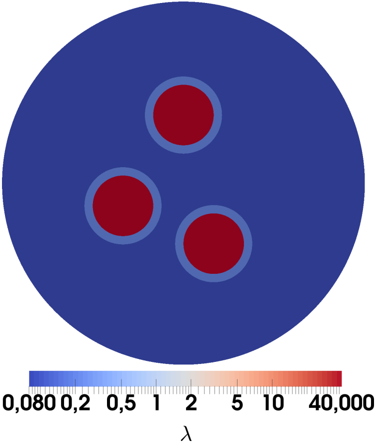
$$



Three cables with the same thermal diffusivity.

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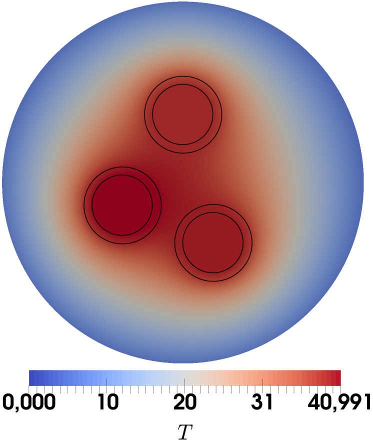

Initial cable positioning and corresponding temperature.

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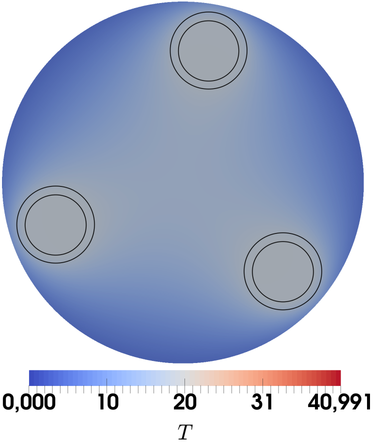


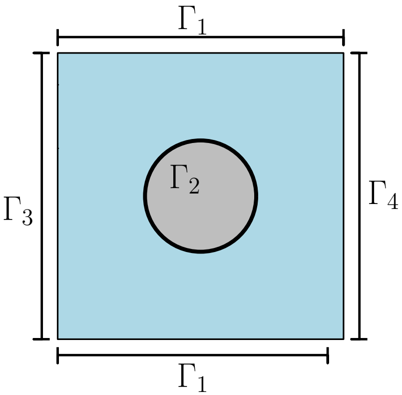
Optimal cable distribution and temperature.

A benchmark problem in shape-optimization is the optimal shape of an obstacle in Stokes-flow

$$\min_{(u,\Omega)} : J(\Omega) = \int_\Omega \sum_{i,j=1}^{2} \left(\frac{\partial u_i}{\partial x_j} \right)^2 \, \mathrm{d}A$$

subject to

$$\begin{aligned} -\Delta u + \nabla p &= 0 \quad \text{in } \Omega, \\ \nabla \cdot u &= 0, \\ u &= 0 \quad \text{on } \Gamma_2, \\ u &= u_0 \quad \text{on } \Gamma_1 \cup \Gamma_3, \\ p &= 0 \quad \text{on } \Gamma_4, \\ C &= C_0, \\ \mathrm{Vol} &= \mathrm{Vol}_0. \end{aligned}$$



Initial setup of the domain.

Olivier Pironneau. "On optimum design in fluid mechanics". In: *Journal of Fluid Mechanics* 64.1 (1974), pp. 97–110.

We achieve the analytical shape, a rugby-ball with a 90 degree front and back angle[5]

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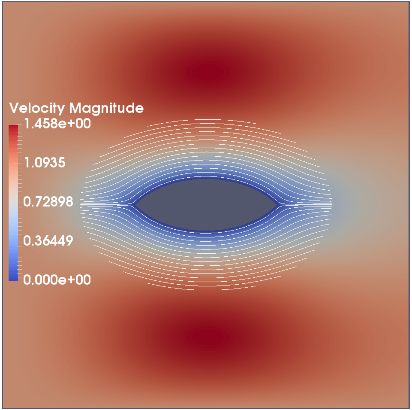

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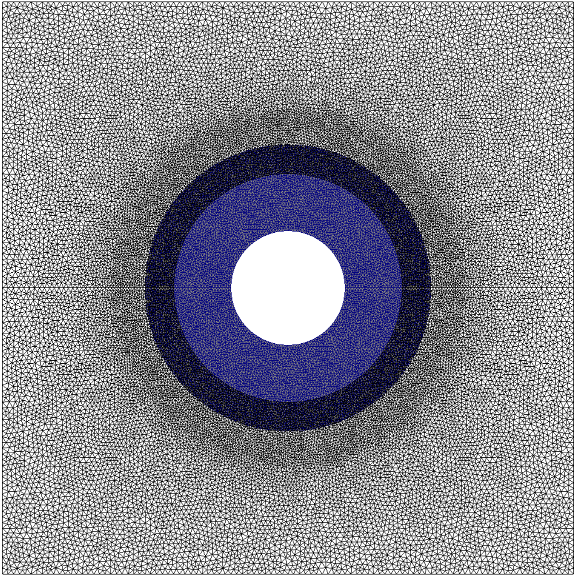
With multiple meshes, we can reduce the size of the mesh that has to be deformed



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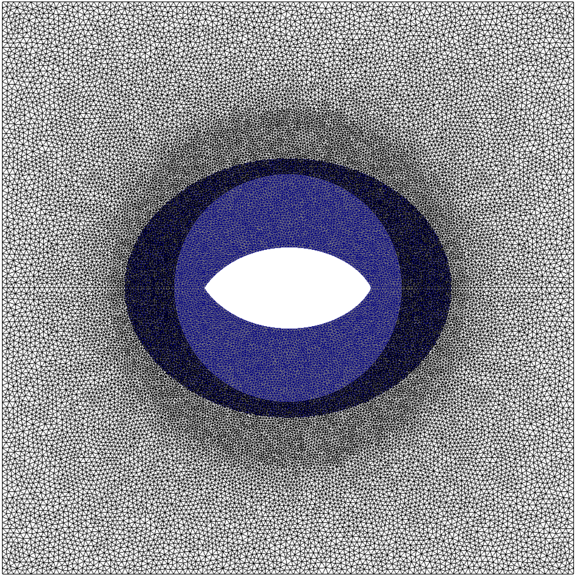

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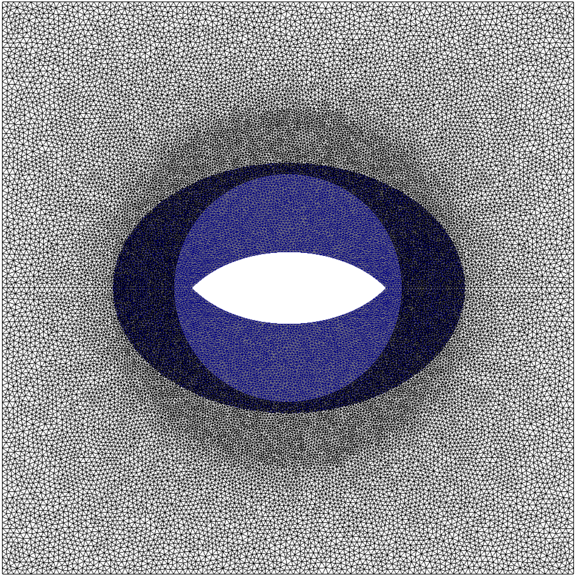
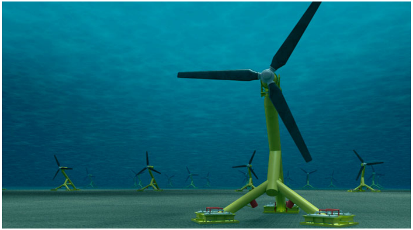


Further work

- Extend the multiple mesh formulation to to time dependent problems such as the NS-equation.
- Use shape-optimization to optimize power-output of a tidal turbine farm.



[islayenergytrust.org.uk/tidal-energy-project/]

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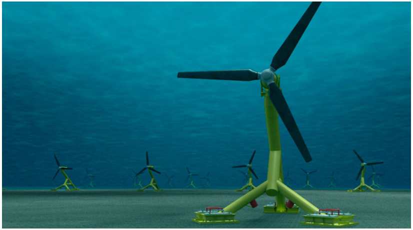

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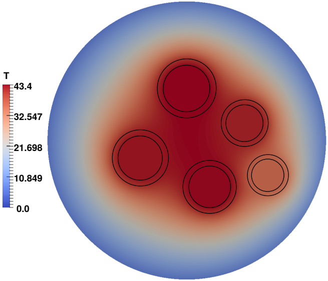
Concluding, FEniCS is currently being extended to employ mixed-domain method and CUT-FEM, where the latter has been used for avoiding re-meshing in shape-optimization



This project is funded by the The Research Council of Norway

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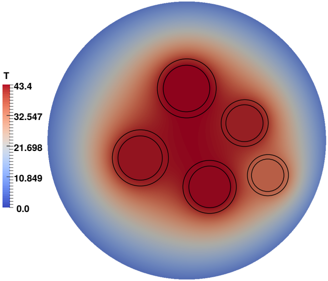
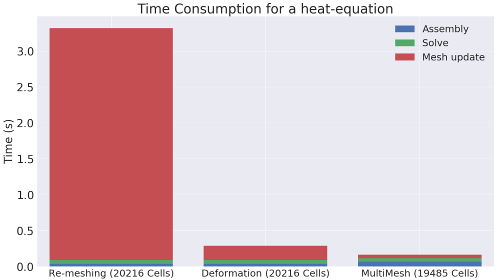
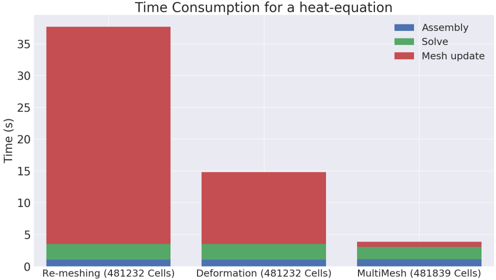


Questions?

This project is funded by the The Research Council of Norway

This trend is clear for both finer and coarser meshes.





The shape-derivative of a functional constrained by PDEs is found with the adjoint method and shape-calculus

$$\min_{\Omega} J(u, \Omega) \text{ s.t. } E(u, \Omega) = 0,$$
$$\hat{J}(\Omega) = J(u(\Omega), \Omega)$$

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Lagrangian based adjoint equation

$$\mathcal{L}(u,\Omega)=J(u,\Omega)+(\lambda,E(u(\Omega),\Omega)).$$
$$\mathrm{d}\hat{J}(\Omega)[s]=\frac{\partial\mathcal{L}}{\partial\Omega}[s]=\frac{\partial J}{\partial\Omega}[s]+\left(\lambda,\frac{\partial E}{\partial\Omega}[s]\right),$$
$$\frac{\partial\mathcal{L}}{\partial u}=\frac{\partial J}{\partial u}[d]+\left(\lambda,\frac{\partial E}{\partial u}[d]\right)=0,\quad\forall d.$$

A linear state equation yields an adjoint equation similar to the state equation

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$$E(u) = Au + b,$$
$$\left(\lambda \frac{\partial E}{\partial u}[d]\right) = (\lambda, Ad).$$

The shape-derivative is transformed into surface integrals with the Hadamard theorem

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$$\mathrm{d}\hat{J}(\Omega)[s] = \frac{\partial \mathcal{L}}{\partial \Omega}[s] = \frac{\partial J}{\partial \Omega}[s] + \left(\lambda, \frac{\partial E}{\partial \Omega}[s]\right),$$

Theorem (Hadamard Theorem)

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Theorem (Hadamard Theorem)

Let $\hat{J}$ be shape differentiable. Then the relation

$$\mathrm{d}\hat{J}(\Omega)[V] = \int_\Gamma \langle V, n \rangle g \ \mathrm{d}S$$

holds for all vector fields.

We consider minimization of the temperature in current-carrying MultiCables as a first example

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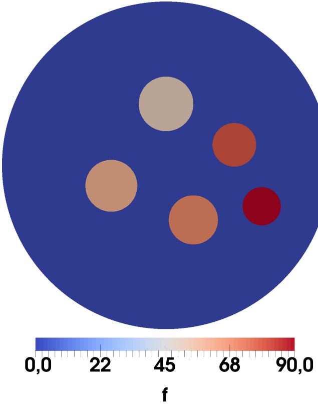

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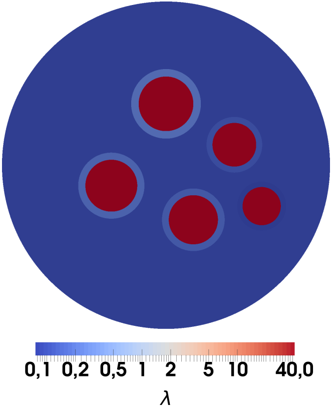
\frac{\partial T}{\partial n} + (T - T_{amb}) &= 0 && \text{on } \Gamma^{ex},\\
[T]_\pm &= 0 && \text{on } \Gamma^1_{int},\\
\left[\lambda \frac{\partial T}{\partial n}\right]_\pm &= 0 && \text{on } \Gamma^1_{int}.
\end{aligned}$$



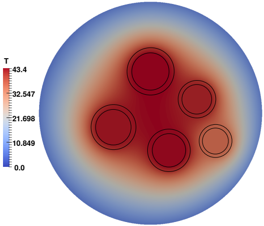
Initial cable positioning and corresponding temperature.

We consider minimization of the temperature in current-carrying MultiCables as a first example

$$\min_{\Omega, T} J(\Omega, T) = \int_\Omega \frac{1}{3}|T|^3 \mathrm{d}x,$$

subject to

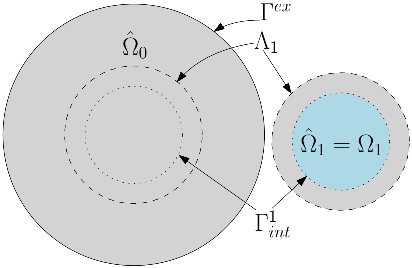
$$\begin{aligned}
-\nabla \cdot (\lambda \nabla T) - cT &= f && \text{in } \Omega, \\
\frac{\partial T}{\partial n} + (T - T_{amb}) &= 0 && \text{on } \Gamma^{ex}, \\
[T]_\pm &= 0 && \text{on } \Gamma^1_{int}, \\
\left[\lambda \frac{\partial T}{\partial n}\right]_\pm &= 0 && \text{on } \Gamma^1_{int}.
\end{aligned}$$



Optimal cable distribution and temperature.

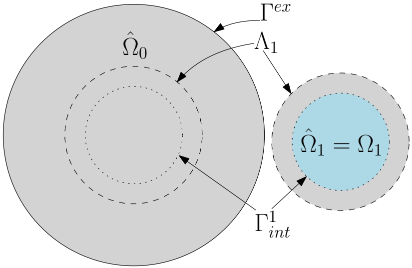
The multiple meshes strong formulation has additional terms for continuity over the artificial interface Λ_1

$$
\begin{aligned}
-\nabla \cdot (\lambda \nabla T) - cT &= f \quad \text{in } \Omega, \\
\lambda_{ex} \frac{\partial T}{\partial n} + (T - T_{ex}) &= 0 \quad \text{on } \Gamma^{ex}, \\
[T]_{\pm} &= 0 \quad \text{on } \Gamma^1_{int}, \\
\left[\lambda \frac{\partial T}{\partial n} \right]_{\pm} &= 0 \quad \text{on } \Gamma^1_{int}.
\end{aligned}
$$



The multiple meshes strong formulation has additional terms for continuity over the artificial interface Λ_1

$$
\begin{aligned}
-\nabla \cdot (\lambda \nabla T_0) - cT_0 &= f \quad \text{in } \Omega_0,\\
-\nabla \cdot (\lambda \nabla T_1) - cT_1 &= f \quad \text{in } \Omega_1,\\
\lambda_{ex}\frac{\partial T_0}{\partial n} + (T_0 - T_{ex}) &= 0 \quad \text{on } \Gamma^{ex},\\
[T]_{\pm} &= 0 \quad \text{on } \Gamma^1_{int},\\
\left[\lambda \frac{\partial T}{\partial n}\right]_{\pm} &= 0 \quad \text{on } \Gamma^1_{int}\\
[u] &= 0 \quad \text{on } \Lambda_1,\\
\left[\frac{\partial u}{\partial n}\right] &= 0 \quad \text{on } \Lambda_1.
\end{aligned}
$$



Nitsches method for weak enforcement of boundary conditions is used to obtain a stable finite element scheme

$$
\begin{aligned}
0 &= F_s(T, v) \\
F_s(T, v) &= \int_\Omega \lambda(\nabla T, \nabla v) - cTv - fv \,\mathrm{d}x \\
&\quad + \int_{\Gamma^{\mathrm{ex}}} (T - T^{\mathrm{ex}})v \,\mathrm{d}s = 0
\end{aligned}
$$

Nitsches method for weak enforcement of boundary conditions is used to obtain a stable finite element scheme

$$
\begin{aligned}
0 &= F_s(T, v) \\
F_s(T, v) &= \sum_{i=0}^{1} \int_{\Omega_i} \lambda(\nabla T, \nabla v) - cTv - fv \,\mathrm{d}x \\
&\quad + \int_{\Gamma^{\mathrm{ex}}} (T_0 - T^{\mathrm{ex}}) v \,\mathrm{d}s = 0
\end{aligned}
$$

Nitsches method for weak enforcement of boundary conditions is used to obtain a stable finite element scheme

$$0 = F_s(T, v) + F_N(T, v)$$

$$F_s(T, v) = \sum_{i=0}^{1} \int_{\Omega_i} \lambda(\nabla T, \nabla v) - cTv - fv \,\mathrm{d}x$$

$$+ \int_{\Gamma^{ex}} (T_0 - T^{ex}) v \,\mathrm{d}s = 0$$

$$F_N(T, v) = -(\langle \mathbf{n}_1 \cdot \nabla T \rangle, [v])_{\Lambda_1} - ([T_h], \langle \mathbf{n}_1 \cdot \nabla v \rangle)_{\Lambda_1} + \frac{\beta}{h}([T], [v])_{\Lambda_1},$$

Nitsche terms	Jump	Average	
$F_N(u, v)$	$[w] = w_1 - w_0$	$\langle w \rangle = \frac{w_1 + w_0}{2}$	

Nitsches method for weak enforcement of boundary conditions is used to obtain a stable finite element scheme

$$0 = F_s(T, v) + F_N(T, v) + F_O(T, v)$$

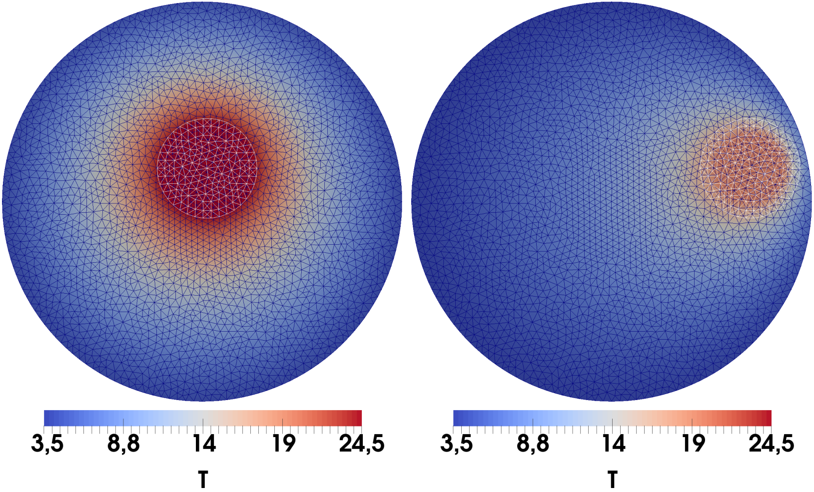
$$F_s(T, v) = \sum_{i=0}^{1} \int_{\Omega_i} \lambda(\nabla T, \nabla v) - cTv - fv \,\mathrm{d}x + \int_{\Gamma^{\mathrm{ex}}} (T_0 - T^{\mathrm{ex}}) v \,\mathrm{d}s = 0$$

$$F_N(T, v) = -(\langle \mathbf{n}_1 \cdot \nabla T \rangle, [v])_{\Lambda_1} - ([T_h], \langle \mathbf{n}_1 \cdot \nabla v \rangle)_{\Lambda_1} + \frac{\beta}{h}([T], [v])_{\Lambda_1},$$

$$F_O(T, v) = ([\lambda \nabla T], [\nabla v])_{\Omega_{h,0} \cap \Omega_1}.$$

Nitsche terms	Jump	Average	Stability on overlap
$F_N(u, v)$	$[w] = w_1 - w_0$	$\langle w \rangle = \frac{w_1 + w_0}{2}$	$F_O(u, v)$

The implementation of the shape-gradient is verified with a Taylor-test

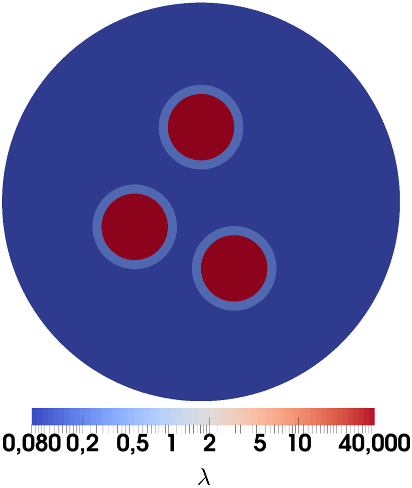


A first example is three internal cables with the same material properties

$$\min_{\Omega, T} J(\Omega, T) = \int_\Omega \frac{1}{3}|T|^3 \mathrm{d}x,$$

subject to

$$\begin{aligned} -\nabla \cdot (\lambda \nabla T) - cT &= f && \text{in } \Omega, \\ \frac{\partial T}{\partial n} + (T - T_{amb}) &= 0 && \text{on } \Gamma^{ex}. \\ [T]_{\pm} &= 0 && \text{on } \Gamma^1_{int}, \\ \left[\lambda \frac{\partial T}{\partial n}\right]_{\pm} &= 0 && \text{on } \Gamma^1_{int} \end{aligned}$$



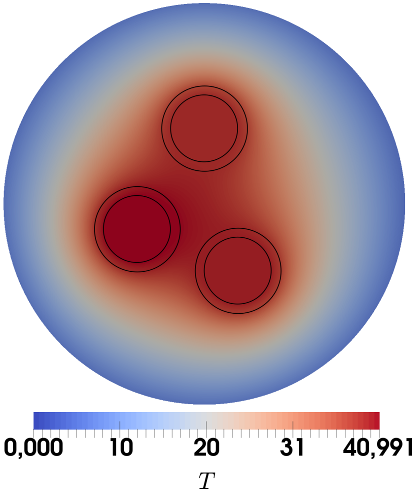
Three cables with the same thermal diffusivity.

A first example is three internal cables with the same material properties

$$\min_{\Omega, T} J(\Omega, T) = \int_{\Omega} \frac{1}{3} |T|^3 \mathrm{d}x,$$

subject to

$$\begin{aligned} -\nabla \cdot (\lambda \nabla T) - cT &= f && \text{in } \Omega, \\ \frac{\partial T}{\partial n} + (T - T_{amb}) &= 0 && \text{on } \Gamma^{ex}. \\ [T]_{\pm} &= 0 && \text{on } \Gamma^{1}_{int}, \\ \left[\lambda \frac{\partial T}{\partial n}\right]_{\pm} &= 0 && \text{on } \Gamma^{1}_{int} \end{aligned}$$



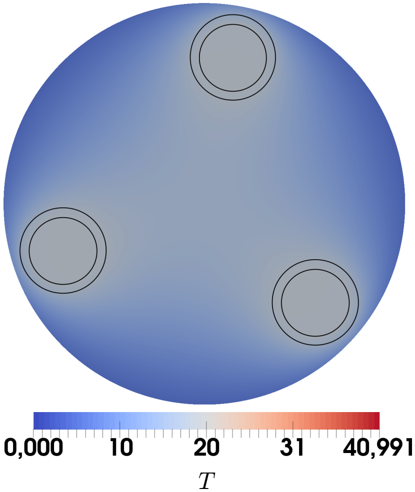
Initial cable positioning and corresponding temperature.

A first example is three internal cables with the same material properties

$$\min_{\Omega, T} J(\Omega, T) = \int_\Omega \frac{1}{3}|T|^3 \mathrm{d}x,$$

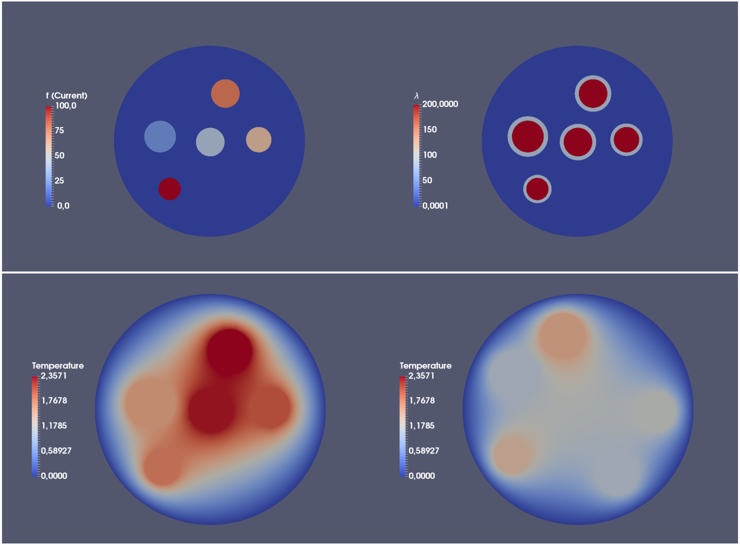
subject to

$$\begin{aligned} -\nabla \cdot (\lambda \nabla T) - cT &= f && \text{in } \Omega, \\ \frac{\partial T}{\partial n} + (T - T_{amb}) &= 0 && \text{on } \Gamma^{ex}. \\ [T]_\pm &= 0 && \text{on } \Gamma^1_{int}, \\ \left[\lambda \frac{\partial T}{\partial n}\right]_\pm &= 0 && \text{on } \Gamma^1_{int} \end{aligned}$$



Optimal cable distribution and temperature.

Optimization of a more complex example with 5 cables with different currents and sizes



The state-equation has been implemented in FEniCS and verified by the method of manufactured solutions

MultiMesh			SingleMesh		
Mesh size	L^2-error	Rate	Mesh size	L^2-error	Rate
0.059	2.41e-04	2.116	0.062	2.27e-04	2.008
0.030	5.92e-05	2.067	0.031	5.65e-05	2.003
0.015	1.46e-05	2.055	0.016	1.41e-05	2.002
0.008	3.68e-06	1.999	0.008	3.52e-06	2.001

Table: Convergence rates of a manufactured Poisson problem. Comparison between MultiMesh and the same mesh described as a single mesh approximated by piece-wise continuous linear elements.

The adjoint system for overlapping meshes has the same stabilization at the artificial interface as the state equation at the interface

$$\sum_{i=0}^{N} \Big((\lambda \nabla p, \nabla v)_{\Omega_i} - (cp, v)_{\Omega_i} \Big)$$

$$+ (\alpha'(T)(T - T^{\mathrm{ex}})p, v)_{\Gamma^{\mathrm{ex}}} + (\alpha(T)p, v)_{\Gamma^{\mathrm{ex}}} = -\sum_{i=0}^{N} (T|T|, v)_{\Omega_i}.$$

Standard terms	Nitsche terms	Overlap terms

The adjoint system for overlapping meshes has the same stabilization at the artificial interface as the state equation at the interface

$$\sum_{i=0}^{N}\Big((\lambda\nabla p,\nabla v)_{\Omega_i}-(cp,v)_{\Omega_i}\Big)$$

$$+(\alpha'(T)(T-T^{\mathrm{ex}})p,v)_{\Gamma^{\mathrm{ex}}}+(\alpha(T)p,v)_{\Gamma^{\mathrm{ex}}}$$

$$+\sum_{i=1}^{N}\Big(-(\langle\lambda\mathbf{n}_i\cdot\nabla p,\rangle[v])_{\Lambda_i}-([p],\langle\lambda\mathbf{n}_i\cdot\nabla v\rangle)_{\Lambda_i}+(\frac{\beta}{h}[p],[v])_{\Lambda_i}\Big)$$

$$=-\sum_{i=0}^{N}(T|T|,v)_{\Omega_i}.$$

Standard terms	Nitsche terms	Overlap terms

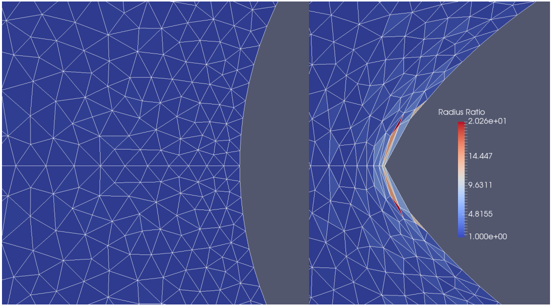
The adjoint system for overlapping meshes has the same stabilization at the artificial interface as the state equation at the interface

$$
\begin{aligned}
&\sum_{i=0}^{N} \Big((\lambda \nabla p, \nabla v)_{\Omega_i} - (cp, v)_{\Omega_i} \Big) \\
&+ (\alpha'(T)(T - T^{\mathrm{ex}})p, v)_{\Gamma^{\mathrm{ex}}} + (\alpha(T)p, v)_{\Gamma^{\mathrm{ex}}} \\
&+ \sum_{i=1}^{N} \Big(- (\langle \lambda \mathbf{n}_i \cdot \nabla p, \rangle [v])_{\Lambda_i} - ([p], \langle \lambda \mathbf{n}_i \cdot \nabla v \rangle)_{\Lambda_i} + (\frac{\beta}{h}[p], [v])_{\Lambda_i} \\
&+ ([\lambda \nabla p], [\nabla v])_{\Omega_{h,0} \cap \Omega_i} \Big) = - \sum_{i=0}^{N} (T|T|, v)_{\Omega_i}.
\end{aligned}
$$

Standard terms	Nitsche terms	Overlap terms

A Laplacian deformation scheme is not suited for large deformations

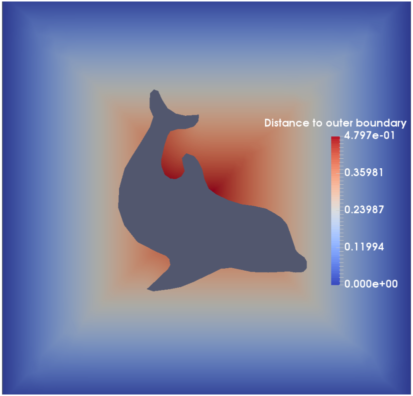
$$
\begin{aligned}
-\Delta w &= 0 \text{ in } \Omega, \\
w &= d \cdot n \text{ on } \Gamma, \\
w &= 0 \text{ on } \partial\Omega \backslash \Gamma.
\end{aligned}
$$



Γ	$d \cdot n$
Moving Boundary	Deformation

The Eikonal convection equation ensures better mesh-quality

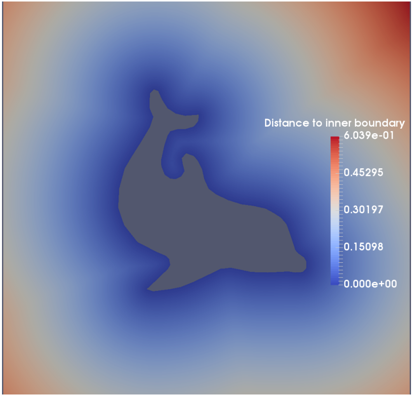
$$-h\Delta\epsilon_1 + ||\nabla\epsilon_1||_2^2 = 1 \text{ in } \Omega,$$
$$\epsilon_1 = 0 \text{ on } \partial\Omega\backslash\Gamma$$



Γ	ϵ_1	
Moving Boundary	Dist. to Γ	

The Eikonal convection equation ensures better mesh-quality

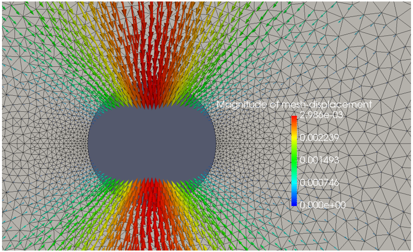
$$-h\Delta\epsilon_1 + ||\nabla\epsilon_1||_2^2 = 1 \text{ in } \Omega,$$
$$\epsilon_1 = 0 \text{ on } \partial\Omega\backslash\Gamma$$
$$-h\Delta\epsilon_2 + ||\nabla\epsilon_2||_2^2 = 1 \text{ in } \Omega,$$
$$\epsilon_2 = 0 \text{ on } \Gamma$$



Γ	ϵ_1	ϵ_2
Moving Boundary	Dist. to Γ	Dist. to $\partial\Omega\backslash\Gamma$.

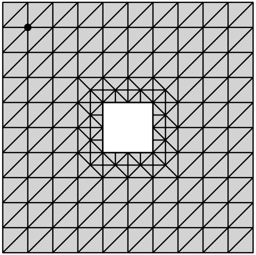
The Eikonal convection equation ensures better mesh-quality

$$
\begin{aligned}
-h\Delta\epsilon_1 + ||\nabla\epsilon_1||_2^2 &= 1 \text{ in } \Omega, \\
\epsilon_1 &= 0 \text{ on } \partial\Omega\backslash\Gamma \\
-h\Delta\epsilon_2 + ||\nabla\epsilon_2||_2^2 &= 1 \text{ in } \Omega, \\
\epsilon_2 &= 0 \text{ on } \Gamma \\
-\alpha\epsilon_2^2\Delta w + \mathrm{div}(\epsilon_1 w \otimes \nabla\epsilon_2) &= 0 \\
w &= d\cdot n \text{ on } \Gamma \\
w &= 0 \text{ on } \partial\Omega\backslash\Gamma.
\end{aligned}
$$



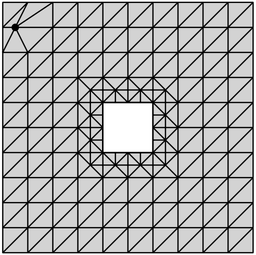
Γ	ϵ_1	ϵ_2
Moving Boundary	Dist. to Γ	Dist. to $\partial\Omega\backslash\Gamma$.

In the discrete case, the solution of a state equation u is dependent of volume nodes



$$J(u(\Omega), \Omega) = \int_{\Omega} u^2 \mathrm{d}\Omega$$

In the discrete case, the solution of a state equation u is dependent of volume nodes



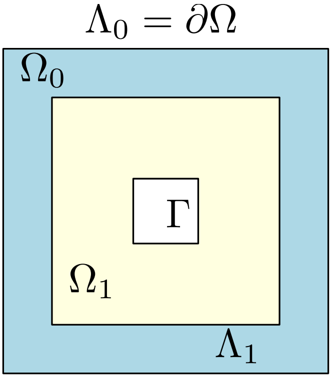
$$J(u(\Omega),\Omega) = \int_\Omega u^2 \mathrm{d}\Omega$$

$$\mathrm{d}J(u(\Omega),\Omega)[V] = \frac{\mathrm{d}}{\mathrm{d}\Omega}\left(\int_\Omega u^2 \mathrm{d}\Omega\right)[V],$$

$$V = \text{Displacement function}$$

The new strong formulation now has additional terms for continuity over the artificial interface Λ_1

$$
\begin{aligned}
-\Delta u_i &= f \text{ in } \Omega_i, \quad i = 0, 1, \\
u_1 + \frac{\partial u_1}{\partial n} &= 1 \text{ on } \Gamma, \\
u_0 + \frac{\partial u_0}{\partial n} &= 1 \text{ on } \partial\Omega \\
[u] &= 0 \text{ on } \Lambda_1, \\
\left[\frac{\partial u}{\partial n}\right] &= \text{ on } \Lambda_1,
\end{aligned}
$$



We need several extra terms to obtain a stable Finite Element scheme

$$0 = a_s(u, v) - l_s(v)$$

$$a_s(u, v) = (\nabla u, \nabla v)_\Omega + (u, v)_{\partial\Omega} + (u, v)_\Gamma,$$

$$l_s(v) = (f, v)_\Omega + (1, v)_{\partial\Omega} + (1, v)_\Gamma$$

We need several extra terms to obtain a stable Finite Element scheme

$$0 = a_s(u, v) - l_s(v)$$

$$a_s(u, v) = \sum_{i=0}^{1} [(\nabla u, \nabla v)_{\Omega_i}] + (u_0, v_0)_{\partial\Omega} + (u_1, v_1)_{\Gamma},$$

$$l_s(v) = \sum_{i=0}^{1} (f, v)_{\Omega_i} + (1, v_0)_{\partial\Omega} + (1, v_1)_{\Gamma}.$$

We need several extra terms to obtain a stable Finite Element scheme

$$0 = a_s(u, v) - l_s(v)$$

$$a_s(u, v) = \sum_{i=0}^{1} [(\nabla u, \nabla v)_{\Omega_i}] + (u_0, v_0)_{\partial\Omega} + (u_1, v_1)_{\Gamma},$$

$$l_s(v) = \sum_{i=0}^{1} (f, v)_{\Omega_i} + (1, v_0)_{\partial\Omega} + (1, v_1)_{\Gamma}.$$

$$a_N(u, v) = -(\langle \mathbf{n}_1 \cdot \nabla u \rangle, [v])_{\Lambda_1} - ([u_h], \langle \mathbf{n}_1 \cdot \nabla v \rangle)_{\Lambda_1} + \frac{\beta}{h}([u], [v])_{\Lambda_1},$$

Nitsche terms	Jump	Average	
$a_N(u, v)$	$[w] = w_1 - w_0$	$\langle w \rangle = \frac{w_1 + w_0}{2}$	

We need several extra terms to obtain a stable Finite Element scheme

$$
\begin{aligned}
0 &= a_s(u, v) - l_s(v) \\
a_s(u, v) &= \sum_{i=0}^{1} \left[(\nabla u, \nabla v)_{\Omega_i} \right] + (u_0, v_0)_{\partial\Omega} + (u_1, v_1)_{\Gamma}, \\
l_s(v) &= \sum_{i=0}^{1} (f, v)_{\Omega_i} + (1, v_0)_{\partial\Omega} + (1, v_1)_{\Gamma}.
\end{aligned}
$$

$$
a_N(u, v) = -(\langle \mathbf{n}_1 \cdot \nabla u \rangle, [v])_{\Lambda_1} - ([u_h], \langle \mathbf{n}_1 \cdot \nabla v \rangle)_{\Lambda_1} + \frac{\beta}{h}([u], [v])_{\Lambda_1},
$$

$$
a_O(u, v) = ([\nabla u], [\nabla v])_{\Omega_{h,0} \cap \Omega_1},
$$

Nitsche terms	Jump	Average	Stability on overlap
$a_N(u, v)$	$[w] = w_1 - w_0$	$\langle w \rangle = \frac{w_1 + w_0}{2}$	$a_O(u, v)$